SIGNAL RECOVERY

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DISCLAIMER

These notes cover the arguments of the course 'Signal Recovery' held by Professor I. Rech at Politecnico di Milano during the academic year 2022-2023.

Since they have been authored by a student, errors and imprecisions can be present.

These notes don't aim at being a substitute for the lectures of Professor Rech, but a simple useful tool for any student (life at PoliMi is already hard as it is, cooperating is nothing but the bare minimum).

Please remember that for a complete understanding of the subject there is no better way than directly attending the course (DIY), which is an approach that I personally suggest to anyone.

In any case, if you found these notes particularly helpful and want to buy me a coffee for the effort, you're more than welcome: https://paypal.me/LucaColomboxc

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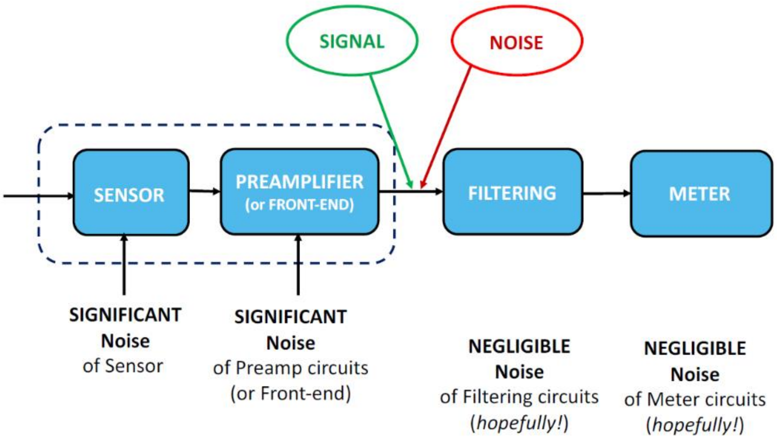
SIGNAL RECOVERY

Normally we have a sensor with noise and an amplifier. The problem is that we cannot change the front end, so we have to deal with noise. So we have the signal and the noise. Just one of the two it is nosense, we will deal with the signal to noise ratio. Our target is to have a SNR equal to 1. This means that the signal and the noise have the same amplitude.



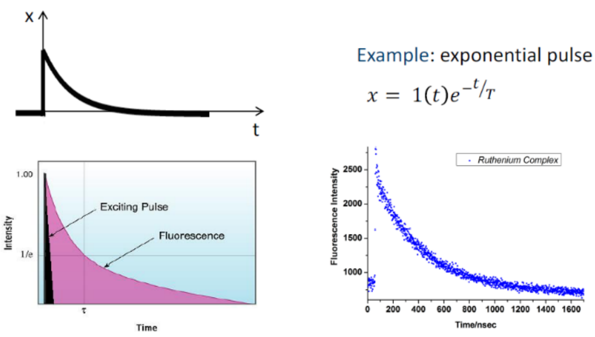
Normally, we want a SNR of at least 3.
We need to include a filtering section, whose noise is typically negligible. We will develop a one stage filter. This because the noise of the amplification is so high that the noise of the filter is negligible.

Finally, we need a meter with negligible noise. It is the instrument that gives us the final measurement. The problem is that we don't know anything about the signal and the noise.
For the signal, every time we will have a different signal, the rect, the exponential and so on, so we cannot study all the possible signals and waveforms, hence we need some tools and properties of the signal that can be applied to all the signals to create filters.

One of the important thing is the time domain and the frequency domain.

MATHEMATICAL DESCRIPTION OF SIGNALS

- **Signals** = electric variables *x* (voltage, current ...) that carry information
- In the domain of time *t* : **deterministic** functions $x = x(t)$



In the domain of frequency *f* (Fourier transform domain) can be considered linear superposition (sum) of elementary sinusoid components

This is a first example of signal, with a laser that excites a fluorescent material. We have an exponential decay. The real signal is the blue one on the bottom right, noisy and with low resolution.

FREQUENCY DOMAIN: RECAP

Signals as linear superposition (sum) of elementary sinusoid components

$$x(t) = \int_{-\infty}^{+\infty} X(f)e^{i2\pi ft}\,df$$

- *X(f)* = Fourier transform of x(t)
- *X(f)* is complex : Module and Phase (or Real and Imaginary parts)

$$X(f) = F[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-i2\pi ft}\,dt$$

The first formula is the formula of a signal in the time domain written as a function of X(f), that is the Fourier transform of the signal. The signal can be expressed as the superposition of waves in the frequency domain. Similar, the signal in the time domain can be written as the superposition of deltas.
In both integral we have the linearity property, which is important because we will work always with linear signals.
As for the Fourier transform, one of its property, starting from the integral, is that in the time domain we will have real signals, while in the frequency domain we have complex signals, so we have both the module and the phase. If I remove the phase (the exponential), from the equations above I need to put t = 0, so that the exponential is 1. So the value of the signal at t = 0 is equal to the integral from -inf to +inf of the Fourier transform in the frequency domain and viceversa.

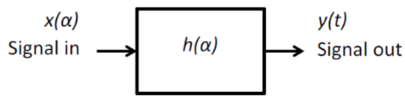
This is important because this will be used a lot of time. For instance, let's integrate from -inf to + inf the Fourier transform of the 1st order LP filter (exponential decay) that is 1/(1+s*tau).
We know the expression in the time domain and, using the equation 1, at t = 0 the integral of the Fourier transform is x(t = 0), which is 1. So the value of the integral is 1.
In the time domain we can consider the signal as an overlap of deltas.

CONVOLUTION: RECAP

Constant-parameter linear filters (NO switches, NO time-variant components!!) are characterized by

H(f) transfer function in frequency domain   *H(f) = F[h(t)]*

h(t) δ-response in time domain   *h(t)* = F^{-1}*H(f)]*



The input x(α) can be described as a **linear superposition** (sum) of elementary **δ-pulses** of amplitude x(α)dα

THEREFORE

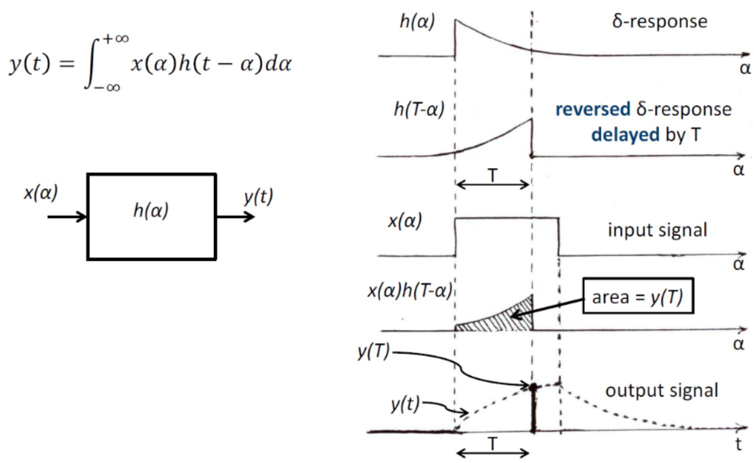
the output *y(t)* can be described as a **linear superposition** (sum) of elementary **δ-pulse responses** *x(α)dα h(t-α)*

$$\boxed{y(t) = x(\alpha) * h(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha}$$

We take the delta of the input, we pass in the filter, we get a delta response. Then we shift the delta response for all the delta of the signal and then we sum at the output all the delta response with the weight of the amplitude of the signal.

Computing the convolution

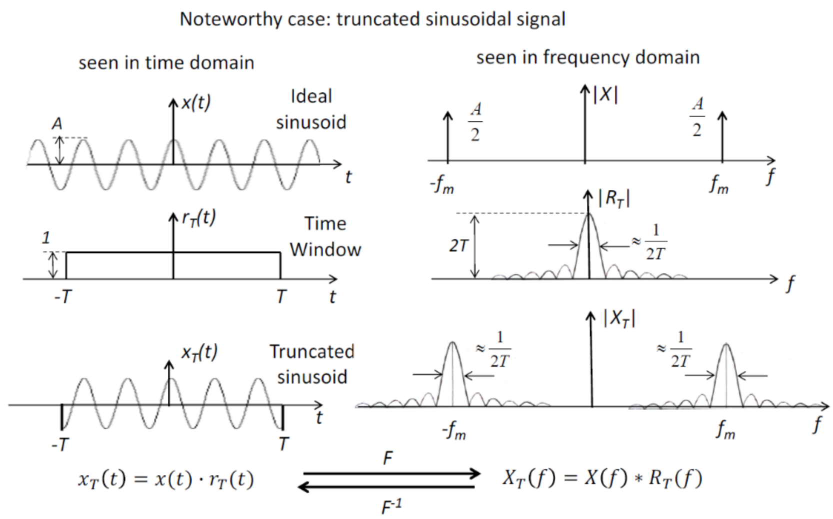
For instance, we need the convolution of the delta response and of my signal.



Top we have the delta response of the filter. To make the convolution we take the delta response, we flip it, we shift the one flipped as a function of the signal and we make the integral of the product.

The convolution of a rect with itself gives as a triangle with a peak equal to the peak value of the rect. If the triangle is the convolution of the rect with itself and the convolution in the time domain is the multiplication in the frequency domain, in the frequency domain we will have a sinh squared.

TRUNCATED SIGNALS



We have a sinusoidal waveform. In the real world it doesn't exist because it goes to -inf to +inf. The Fourier transform of the sinusoidal waveform is two delta of half amplitude at the frequency of the sinusoidal. It must be remembered.
I need to multiply the sinusoidal by a rect, whose Fourier transform is the sinh, I truncate the sinusoid in the time domain and so in the frequency domain I'm making the convolution of the two deltas and the sinh. This is the Fourier transform of a truncated signal.

It is important for the following reasons. In real world, we see just a part of the sinusoid.
To avoid aliasing, we need a sampling frequency that is two times the maximum frequency of the signal. If we have a truncated signal, which is the frequency of the sinusoid? We are creating aliasing because in the frequency domain the truncated sinusoid has the sinh that goes to -inf and +inf in the spectrum.

- In reality the signal is always available over a finite time interval: therefore, **in reality we always deal with truncated signals**
- cropping in time corresponds to **convolution** of the signal in the *f* domain with the transform of the rectangle (*sinc* function)
- the convolution spreads the signal in the *f* domain; that is, it makes it wider and smoother
- the narrower is the window *2T*, the wider is the *sinc* and more significant is the signal spreading in frequency
- Applying correctly the sampling theorem we see that: the **sampling frequency f_S** to be employed for a **truncated** sinusoid of frequency f_m is **NOT** $f_S \approx 2f_m$ **;** it must be REMARKABLY HIGHER $f_S >> 2f_m$

SIGNAL ENERGY

From a signal we can recover some information, e.g. the energy of the signal. We need to use all they possible information to increase the SNR, but at the same time simplifying them.
The energy is defined as in the image.

The Energy *E* of a signal *x(t)* is defined as

$$E = \lim_{T\to\infty} \int_{-T}^{T} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$$

Signals *x(t)* with finite *E* are called **energy-signals**. Typical example: **pulse signals**

INTUITIVE VIEW OF ENERGY:

Let x(t) be a voltage pulse on a unitary resistance R=1 Ω,

Power=V^2/R then E is the energy dissipated in R by the pulse

EXAMPLE

It is the integral of the square of the signal in the time domain from -inf to +inf. For instance, let's take the voltage over a resistor, this integral is exactly the energy on the resistor from the electrical point of view.

The other important formula is the **autocorrelation formula**.

Signal Auto-Correlation Function (Energy-type)

$$k_{xx}(\tau) = \lim_{T\to\infty}\int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

$k_{xx}(\tau)$ gives the **degree of similarity** of $x(t)$ with itself **shifted by τ**

Energy = Autocorrelation at zero-shift

$$\boxed{k_{xx}(0) = E}$$

It is the integral from -inf to + inf of what in the image. It is similar to the convolution but not the same. This formula gives us the level of similarity of the function (or signal) with itself. In the previous image we defined the energy. <u>Energy is nothing else than the autocorrelation at zero shift</u>.

Example

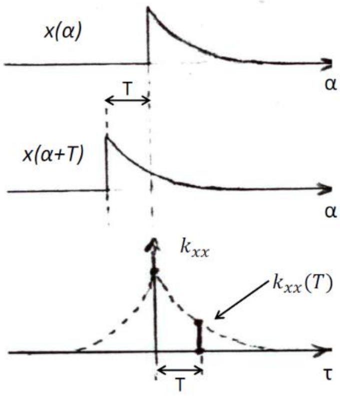
$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Case: exponential decay

$$x = 1(t)Ae^{-t/T_P}$$

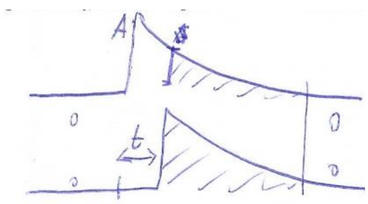
tau

$$k_{xx}(\tau) = A^2\frac{T_P}{2}e^{-\tau/T_P}$$

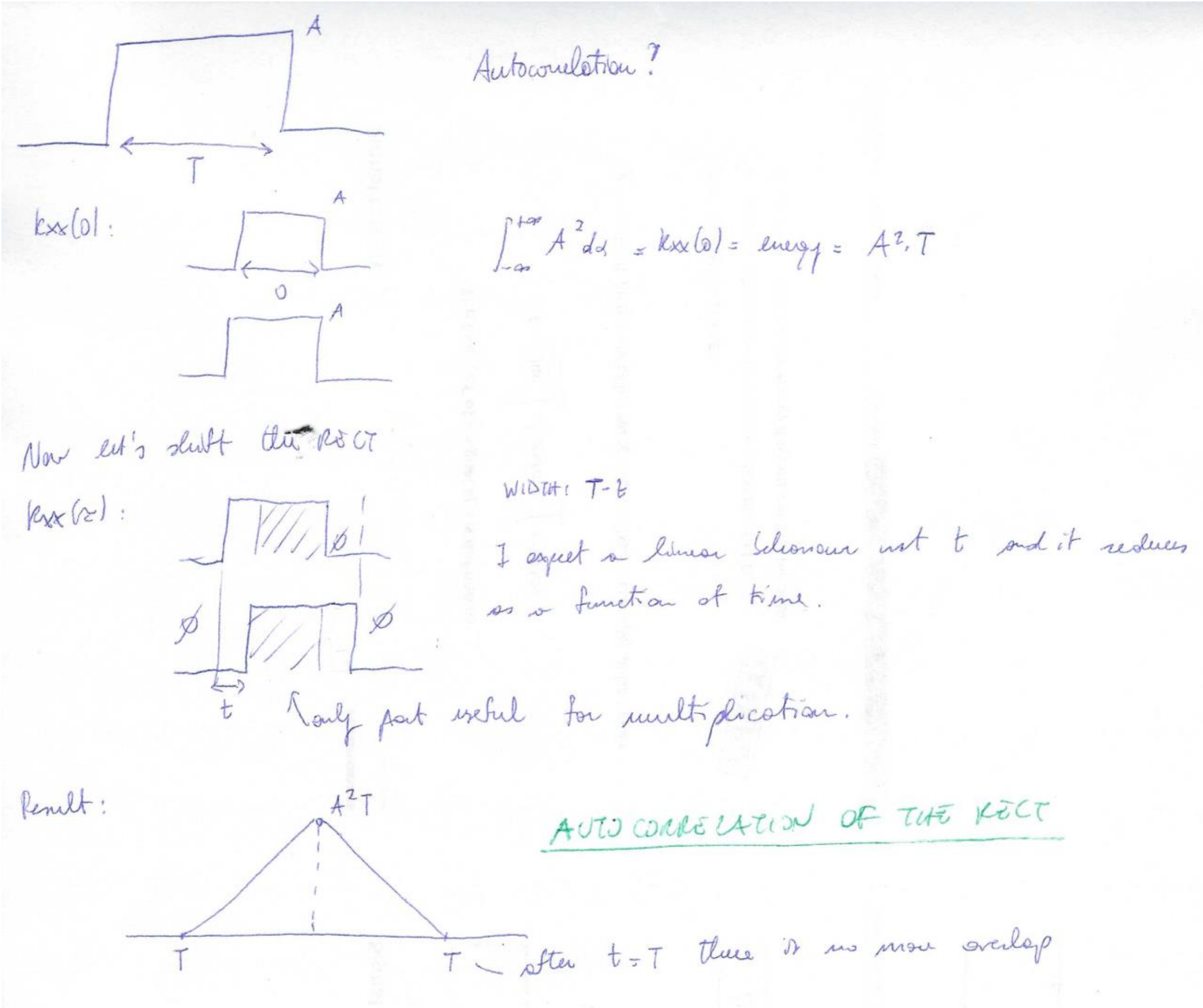


Some useful signals are the exponential decay time and the rect. Which is the autocorrelation of the exponential decay time? I take the exponential decay time and the exponential decay time with the shift equal to zero, we multiply them and make the integral. So we add a factor 2 to the exponential. So we get an exponential whose tau is half the original tau of the signal if I put the 2 at the denominator. Thus the integral will be half of the tau (integral of the signal was tau). So the energy of the signal is tau/2.

Now I shift the signal and make the same computation, so multiply the two signals and make the integral. The multiplication of the two functions at the extremes is 0 (both are zero), and in general if one of the two signals is zero the multiplication is 0. So we need just the integral of the part where both are different from zero. The first signal has the same original tau, but a smaller amplitude.

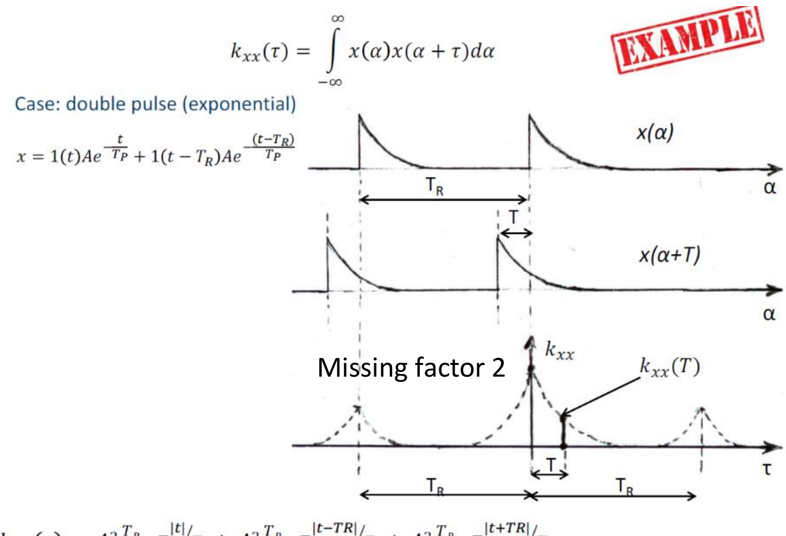


Another example, the **autocorrelation of the rect**.



Autocorrelation of a delayed exponential is the following.

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$



$$k_{xx}(\tau) = A^2\frac{T_P}{2}e^{-|t|/T_P} + A^2\frac{T_P}{2}e^{-|t-TR|/T_P} + A^2\frac{T_P}{2}e^{-|t+TR|/T_P}$$

CROSS CORRELATION

$$k_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} x(\alpha)y(\alpha+\tau)d\alpha = \int_{-\infty}^{\infty} x(\alpha)y(\alpha+\tau)d\alpha$$

- x(t) and y(t) are **two different** signals of energy-type
- ***$k_{xy}(\tau)$* gives the degree of similarity of *x(t)* with *y(t)* shifted by τ to left (towards earlier time)**
- Various denominations for $k_{xy}(\tau)$:
  Cross-Correlation function of x and y
  Mutual-Correlation function of x and y

Level of similarity between two different signals. It is the same of the autocorrelation but the second signal is shifted.
The formula is similar to the one of the convolution. We take the signal, the second signal, we shift, multiply and make the integral. It seems the convolution but without the flipping that we have in the convolution.
If I make the convolution of signals x and y I get z(T).

Convolution

$x * y = z(T)$

is different from Crosscorrelation $k_{xy}(T)$

However

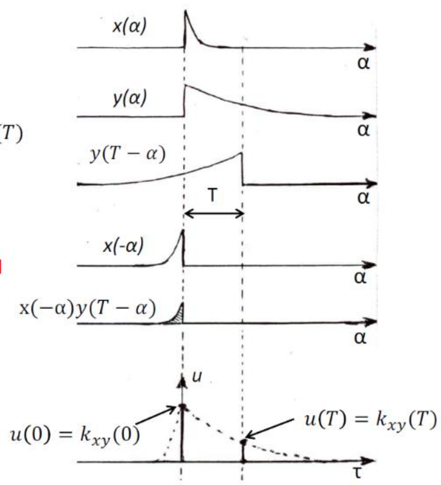
Convolution with **first term reversed**

$x(-a) * y(a) = u(T)$

is equal to Crosscorrelation

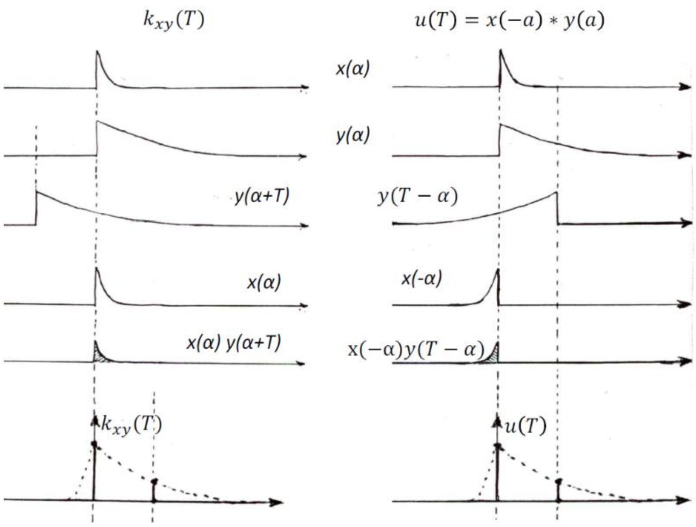
$u(T) = k_{xy}(T)$

$$\boxed{k_{xy}(T) = x(-a) * y(a)}$$



Now I do the convolution with a modification, instead of x(alpha) I take x(-alpha) and do the convolution. Now x and y are in the same direction, so I'm doing the cross correlation. So <u>if I take the first signal, I flip it and I convolve with the second one I'm doing the cross correlation</u> (because in the convolution I would have need to flip the second signal, so I return back with the two signals in the same direction).

The last formula in the box is important because we don't know anything for the cross correlation, but we know that the convolution in the time domain is the multiplication in the frequency domain, we have a connection for the cross correlation in the frequency domain. So we have the possibility to switch from the frequency domain to the time domain.



ENERGY SPECTRUM

Energy signal $x(\alpha)$ with Fourier transform X(f): by Parseval's theorem

$$\boxed{E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha = \int_{-\infty}^{\infty} |X(f)|^2 df = 2\int_0^{\infty} |X(f)|^2 df}$$

$S_x(f) = |X(f)|^2$ is called the **Energy Spectrum** of the signal $x(\alpha)$

INTUITIVE VIEW OF ENERGY SPECTRUM:

(1) Let $x(t)$ be voltage on a unitary resistance $R=1\,\Omega$

(2) $x(t)$ = sum of sinusoid components with frequency f and amplitude $|X(f)|df$

(3) Sinusoids are orthogonal functions : **No power** from multiplication of **different components** (different f)

Every *component (at frequency f) contributes an energy:*

$$\boxed{dE = 2\,|X(f)|^2 df}$$

EXAMPLE

It gives an information on how the energy of the signal is distributed.

We have just to use the formula in the first box, which is the Perceval's theorem. It applies to real signal. The energy is defined as the integral of x^2 from -inf to +inf. The Perceval theorem tells us that we can make the integral of the square of the signal in the time domain (if real) and getting the same result if we make the integral of the square modulus of the Fourier transform, both from -inf to +inf.
In the time domain I have the energy and I want to have some information related to the energy also in the frequency domain, and this can be done with this relationship.

I define the square modulus of the Fourier transform as the energy spectrum. It is something that integrated from -inf to + inf in the frequency domain gives us the energy.

In the real world, the spectrum is not correct just from -inf to +inf, but also for a small amount of frequency. I can evaluate the spectrum at a specific frequency with a small df.

- Alternative definition of the **Energy Spectrum** is

$$\boxed{S_x = F[k_{xx}]}$$

- Knowing that $k_{xx} = x(-\alpha) * x(\alpha)$ we see that the two definitions are consistent

$$S_x = F[k_{xx}] = F[x(-\alpha) * x(\alpha)] = X(-f)X(f) = X^*(f)X(f) = |X(f)|^2$$

and by a basic property of Fourier transforms

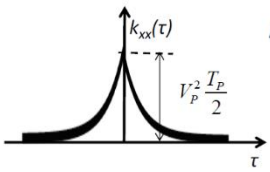
$$\boxed{E = k_{xx}(0) = \int_{-\infty}^{\infty} S_x(f)df = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

The energy spectrum is also the Fourier transform of the autocorrelation. Then the energy is the autocorrelation in zero that is the integral of the square modulus of the Fourier transform.
The spectrum is indeed a function that, if integrated in the frequency domain from -inf to +inf gives us the energy. But it is also the Fourier transform of the autocorrelation.

The integral of -inf to +inf of a function in the frequency domain is the value in zero of the inverse Fourier transform. So the inverse Fourier transform of the spectrum has to be the value in zero, so the energy, that is the autocorrelation in zero.

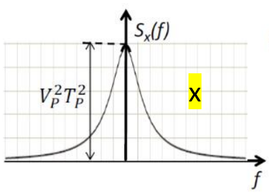
The tau of the exponential is Tp. We know the autocorrelation function (it is tau and not |t|). We also know the energy, which is the value in 0 of the autocorrelation.
Then we want to move in the frequency domain, and we get the spectrum. Plot x is a linear-linear plot, not the Bode diagram. In the linear-linear plot, the area of the plot will give us the value of the white noise of the signal, so to compare the filters we need just to compare the area. Same reasoning for the 1/f noise but with a linear-log plot.

Exponential pulse: $x(t) = V_P e^{-t/T_P}$ $\quad X(f) = V_P T_P \frac{1}{1+j2\pi f T_P}$



Autocorrelation function: $k_{xx}(\tau) = V^2 \frac{T_P}{2} e^{-|t|/T_P}$

Energy: $E = k_{xx}(0) = V^2 \frac{T_P}{2}$



Energy Spectrum: $S_x(f) = |X(f)|^2 = V^2{}_P T^2{}_P \frac{1}{1+(2\pi f T_P)^2}$

Energy: $E = \int_{-\infty}^{\infty} S_x(f)df = V^2 \frac{T_P}{2}$

SIGNAL POWER

The energy of a sinusoidal waveform is infinite, the average value is zero but the energy not. So instead of the energy we can define the power. All that was defined for energy has to be defined for power.
The power is the limit of the square of the signal divided by two times the period 2T (and we added the limit). It has the same shape of the energy formula.

For signals *x(t)* that have NOT finite energy $E \to \infty$ (DC, sinusoids, periodic signals, etc.)
the **Power *P*** is defined as the time-average

$$P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$$

Parseval theorem is valid for the entire integral $\int_{-\infty}^{+\infty}$
but NOT for the truncated integral $\int_{-T}^{+T}$
For computing *P* in *f* domain instead of truncated integral we use truncated signal $x_T(t)$

$$x_T(\alpha) = x(\alpha) \quad \text{for } |\alpha| \le T$$
$$x_T(\alpha) = 0 \quad \text{for } |\alpha| > T$$

We can thus exploit Parseval theorem: with $X_T(f) = F[x_T(\alpha)]$ we get

$$P = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{{x_T}^2(\alpha)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} df \quad \text{x}$$

The Power Spectrum of the signal *x(α)* is defined as the integrand

$$S_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T} \quad \text{and} \quad P = \int_{-\infty}^{\infty} S_x(f) df$$

We are considering power signals, not energy signals. If instead of x we use the truncated signal x_T, at this point we can write the integral from -inf to +inf, because it is zero where we are not interested.
Once we have the truncated signal and the integral from -inf to +inf we can use the Parseval theorem.

In x the limit is shifted inside because the integral is linear. So also for the spectrum definition we have something similar, but we add the limit and 2T.

Auto-correlation function – power type

Just like power P, the autocorrelation of power signals is defined as **time-average**

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha \qquad \text{note that} \qquad P = K_{xx}(0)$$

Also here we use truncated signal $x_T(\alpha)$ instead of truncated integral $\int_{-T}^{T}$

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)x_T(\alpha+\tau)}{2T} d\alpha$$

NB1: for finite T it is $\int_{-T}^{T} x(\alpha)x(\alpha+\tau)d\alpha \neq \int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha$
but for $\lim_{T\to\infty}$ the = is valid

The autocorrelation in zero is exactly the power, so the formula is correct (equivalent seen for the energy).

The autocorrelation of the signal is the limit of the integral from -T to T of the formula in the second line. Now I use the same trick as before shifting to -inf to +inf the integral and with the truncated signal. The thing is that the equation, without the limit, is not valid. If I have tau = 0 the equation is correct for any t, but if tau != 0, the equation is equal only for the limit that goes to inf.

$$\boxed{K_{xx}(\tau) = \lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}}$$

An alternative definition of signal <u>Power Spectrum</u> is

$$\boxed{S_x = F[K_{xx}(\tau)]}$$ x

The two definitions are consistent

$$S_x(f) = F[K_{xx}(\tau)] = F[\lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}] = \lim_{T\to\infty} \frac{F[k_{xx,T}(\tau)]}{2T} = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

and

$$\boxed{P = K_{xx}(0) = \int_{-\infty}^{\infty} S_x(f)df}$$

So the autocorrelation is the limit of the autocorrelation of the truncated signal (bottom image, first formula). The autocorrelation of the truncated signal is known because the truncated signal is an energy signal because the truncated signal is limited in time.
Hence we can define the new power spectrum as the Fourier transform of the autocorrelation (x). We have the exact same relationships that we had for energy. Of course, inside the formula of the spectrum and so on we have the limit, but since we will never calculate the integral, we are good.

Cross-correlation function – power type

$$K_{xy}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)y(\alpha+\tau)}{2T} d\alpha = \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T(\alpha)y_T(\alpha+\tau)}{2T} d\alpha$$

x(t) and y(t) are two different signals, both power-type

$K_{xy}(\tau)$ measures the degree of similarity of *x(t)* with *y(t)* shifted by τ to left (towards earlier time)

If even only one of the two signals x(t) and y(t) is energy-type
the energy type autocorrelation $k_{xy}(\tau)$ must be employed

(in fact, the power-type crosscorrelation vanishes $K_{xy}(\tau) = 0$
and the energy-type crosscorrelation $k_{xy}(\tau)$ is finite).

If one of the signal is power and the other one is energy, what do we do? The energy type autocorrelation must be used.

Comparison between energy and power

Energy-type (pulses etc.)

Energy $E = \int_{-\infty}^{\infty} x^2(\alpha)d\alpha$

Autocorrelation

$$k_{xx}(\tau) = \int_{-\infty}^{\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

Energy spectrum

$$S_{x,e} = F[k_{xx}(\tau)] = |X(f)|^2$$

and

$$E = \int_{-\infty}^{\infty} S_{x,e}(f)df$$

Power-type (periodic waveforms etc.)

Power $P = \lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha$

Autocorrelation

$$K_{xx}(\tau) = \lim_{T\to\infty} \int_{-T}^{T} \frac{x(\alpha)x(\alpha+\tau)}{2T} d\alpha$$

Power spectrum

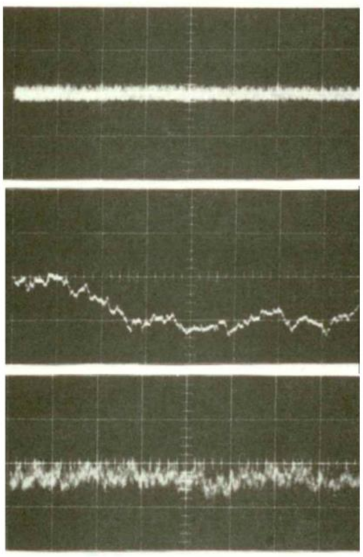
$$S_{x,p} = F[K_{xx}(\tau)] = \lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}$$

and

$$P = \int_{-\infty}^{\infty} S_{x,p}(f)df$$

NOISE

NOISE WAVEFORMS



White Noise

spectrum $\boldsymbol{S}$ = ***constant***

Random-Walk Noise

spectrum $\boldsymbol{S} = \frac{1}{f^2}$

Flicker Noise

spectrum $\boldsymbol{S} = \frac{1}{f}$

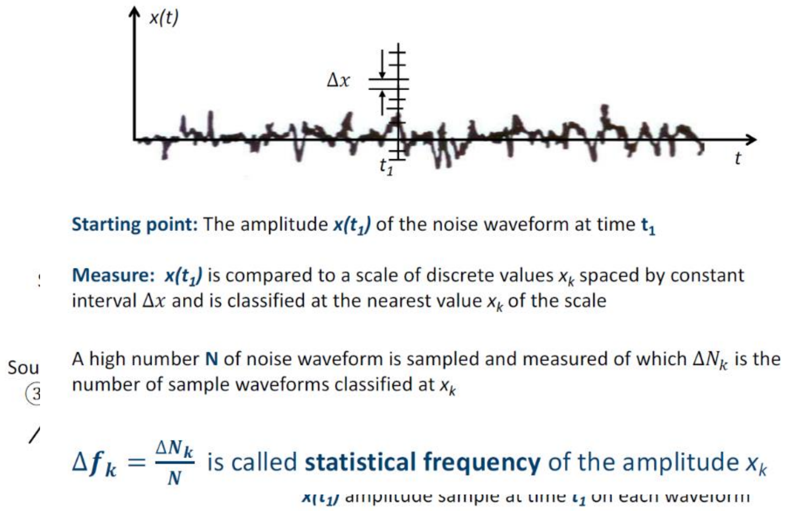
The problem with noise is that we have more **ensembles**. The case is the image below.

The ensemble tells us that if we consider 3 identical amplifier and we apply the same input signal, at the output we will have the same output signal, but for the noise, these 3 identical amplifiers will have three different waveforms. So we need to introduce the fact that the noise signal is no more deterministic.

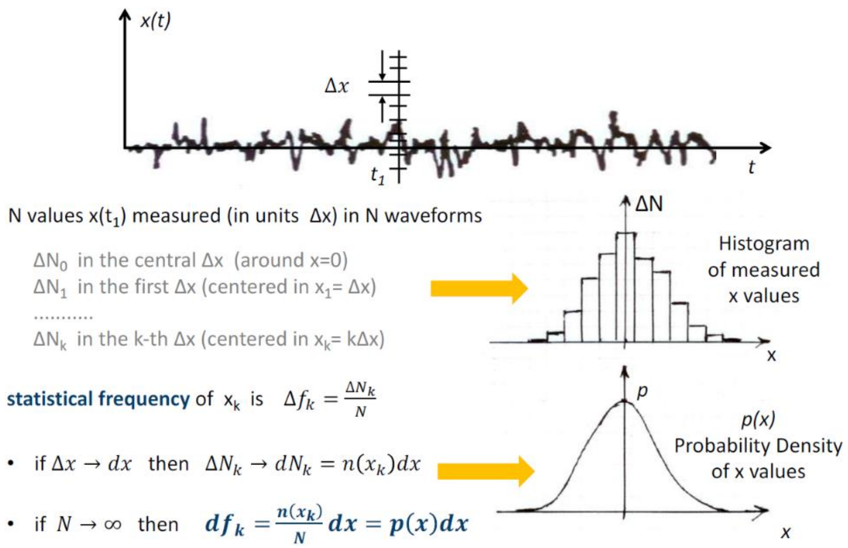
Classifying the amplitude of noise samples

Let's focus on t1. We choose one time of the axis and we study what happens for one waveform. So at t1 we have one value for each replica, so if we have thousands of replica we have thousands of values. What I can do is to see how many times the amplitude of the replicas has a certain amplitude at t1, I count them. Then I divide the number of times I found by the number of replicas (e.g. in how many replicas the signal is in the second step? In the second one? Then I count the number of times in the different replica).

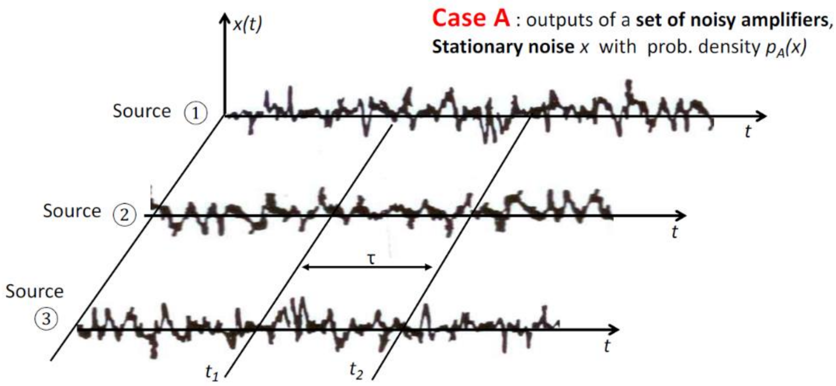
If I make the plot I get an histogram.



The distribution I get is a statistical distribution. The distribution I get is the probability. The integral of the probability function is 1. So I have the probability to have a certain value.



The problem is that this result is not enough, we have to replicate the information for every time. We are lucky because sometimes the noise is stationary, i.e. the distribution is the same for every time instant. But also in this case we still don't have a complete description of the noise, the probability is not enough. We can demonstrate that the probability is not enough. To demonstrate, let's consider three different amplifiers and for every time someone has given us the probability density. It is enough?
If I consider t1 and t2, we are considering a real amplifier, and if the distance between t1 and t2 is small, the voltage is similar. But if the time distance is different, there is no way they are similar. So when the time difference is very small, there will be a strong correlation between the two values, while if tau increases the correlation drops.

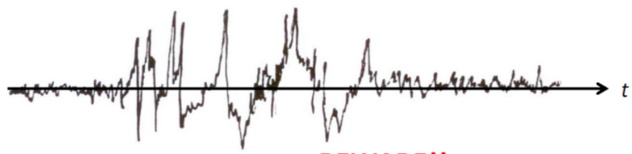


Values $x(t_1)$ and $x(t_2)$ measured on a sample waveform at different t_1 and t_2 are random values with equal probability density $p_A(x)$ and they are:

- in practice **identical for ultra-short** interval τ
- somewhat **different for short** interval τ
- different and **independent for longer** interval τ

NON-STATIONARY noise :

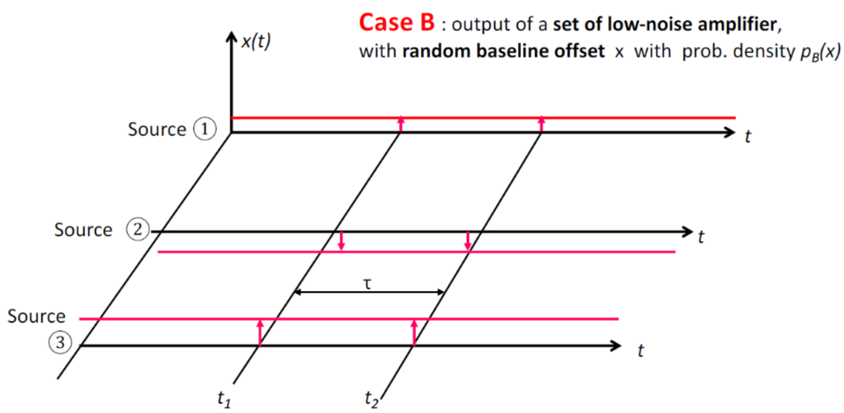
the **probability density varies** in time $p = p(x, t)$



BEWARE!!

the probability density ***p* alone does not** give a complete description of the noise, in fact different cases can have equal probability density *p*

Now we consider another example with the same probability density in t1 and t2. If the probability density is enough to describe the noise we would obtain the same result. This time, instead of the output



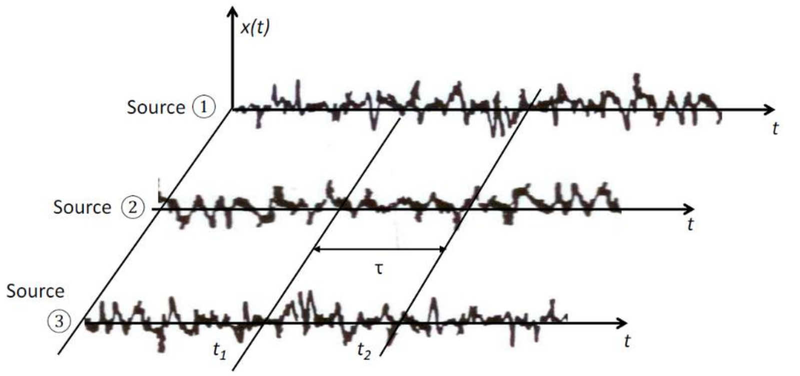
Values $x(t_1)$ and $x(t_2)$ measured on a sample waveform at different t_1 and t_2 :

- they are random values with probability density $p_B(x)$;
- they are equal for any interval τ , short or long

Case B is **different** from A, but it can have **equal probability density $p_B(x) = p_A(x)$**

of the amplifier we take the offset. The offset is not deterministic, it changes from one amplifier to the other. At t1 I will have an offset different for each amplifier, I cannot predict it, it is different for any amplifier. Then I take t2. The offset is constant in the amplifier, we don't know it but it is a number. Hence as soon as we have the offset at t1, it is the same for any time. It is different for the replicas but the correlation between t1 and t2 of the same replica is 1 for all the time. The problem is that the probability density in each point is equal to the probability density of the previous point. Hence just the probability density is not enough to describe our problem.

COMPLETE DESCRIPTION OF THE NOISE

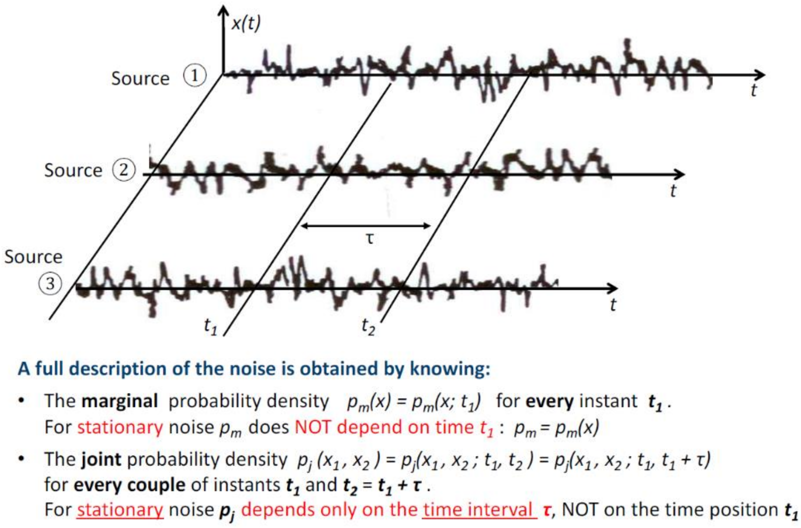


- For a proper description of the noise the **marginal** probability $p_m(x, t)dx$ of having a value x at time t is **NOT sufficient**
- The **joint** probability $p_j(x_1, x_2, t_1, t_2)dx_1\, dx_2$ of having a value x_1 at time t_1 **and** a value x_2 at time t_2 **must also be considered**

We have infinite ensembkes and, for each replica we have the time and amplitude axes. We can define the probability density for each time, which is the probaibilty to have a certain value at a certain time, but it is not enough, so we need the joint probability, which is the probability of having a certain value at time t2 if I have a certain vale at time t1. So it is a function of x1, x2, t1 and t2 (x1 and x2 are amplitude values).

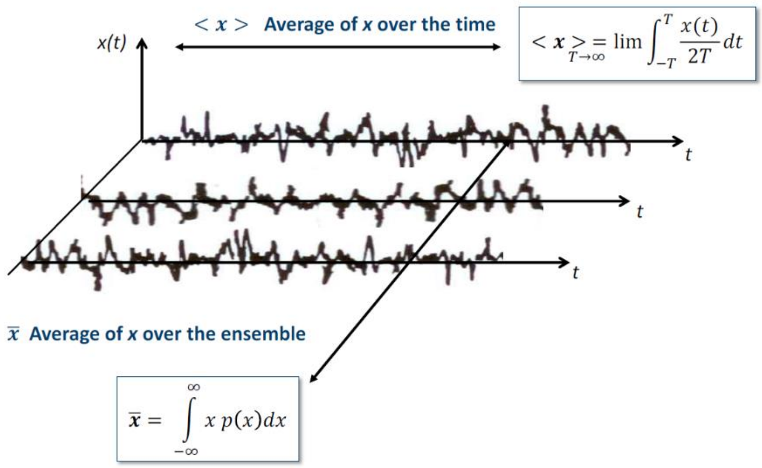
The problem is that the formula is vary complicated to write and manage.
So our description of noise is composed by **marginal probability** (probability having a certain value at a ceartain time), and for stationary noise the probability density doesn't depend on t, it is the same for any time (definition of stationary).

Then we need the **joint probability** because the marginal one is not enough. Again, if the noise is stationary, so the marginal probability is the same for all the times, it is possible to demonstrate that the joint probability depends only on the time interval tau, and not on t1 and t2, we are interested just in the distance between them.



Time average and ensemble average

We can do a thing we did with the signal. We try to extract some numbers that are summary of some properties. E.g. the energy of the signal or the shape of the signal, which are two different things. With noise we have two directions, time and ensembles, so the complexity is squared.
Since we have two direcitons, we can try ti simplify in both.
The upper formula in the image is similar to the one of the energy, but we don't have the square, we are just averaging over time the signal. The average in time is indicated with <x>.



We have also the ensambles, so we can try to extract also the average on ensambles. We are at one time, and at this time we have the probability of having a certain value. I sum all the values and divide by the number of values, but the problem is that in the ensemble direction the system is dominated by the statistic, so we cannot sum all the possible values, because we have an infinite number of replicas. So the idea is that we go from -inf to +inf of the probability. So **we are making an average of the probability density**.

DESCRIPTION OF NOISE WITH 2ND ORDER MOMENTS OF PROBABILITY DISTRIBUTION

Normally, the average over time of the noise is zero, and the average on the ensembles is zero.

Noise is different from disturb and from offset. In general, the background like an offset of the noise is not a problem, because it is deterministic and we can remove it. The problem is that associated to the background there is noise, but the background itself is not a problem, we can measure it and remove it.

So we want something different than the averages on ensembles and time. We need the moments.
The moment of the marginal probability is the integral from -int to +inf of the x^n times the probability. It is the same as before, but we increase the power, so the average is the moment of order 1. The same can also be done for the joint probability.

***NB:** for clarity, we call here the two statistical variables x and y instead of x_1 and x_2*

Moments of a marginal *p(x)* $\quad m_n = \overline{x^n} = \int_{-\infty}^{\infty} x^n\, p(x) dx$

Moments of a joint *p(x,y)* $\quad m_{jk} = \overline{x^j\, y^k} = \int_{-\infty}^{\infty} x^j\, y^k p(x,y) dxdy$

- the m_n (and m_{jk}) give information on the features of the distributions
- as the order (*n* or *j+k*) increases, the information is increasingly of detail

Let's consider a description of noise limited to the 2° order moments, i.e.

Mean square value (or variance)

$$m_2 = \overline{x^2} = \int_{-\infty}^{\infty} x^2\, p(x) dx = \sigma_x{}^2$$

Mean product value (or covariance of *x* and *y*)

$$m_{11} = \overline{xy} = \int_{-\infty}^{\infty} xy\, p(x,y) dxdy = \sigma_{xy}{}^2$$

NB: it is obviously

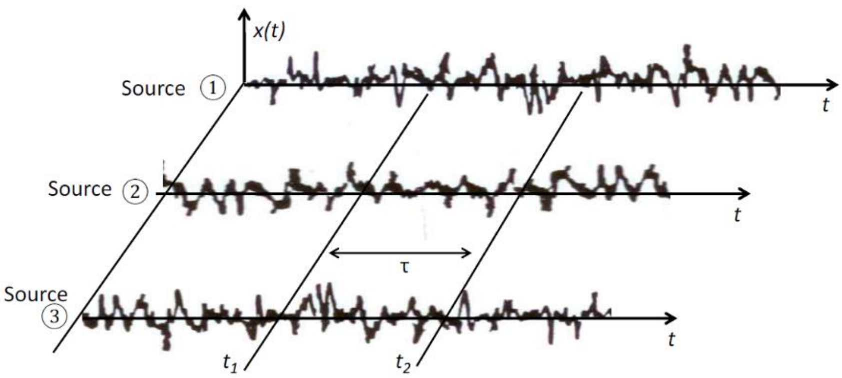
$m_o = m_{oo} = 1$ the total probability is normalized to 1

$m_1 = m_{1o} = \bar{x} = 0 = \bar{y} = m_{01}$ the mean value of noise is zero

As soon as n increase, we increase the detail we can observe. We will stop at n = 2 so the average and the moments of the second order.
n = 0 is 1, because it is the integral of the probability, so it is useless.

The mean square value is the moment of the second order, but if we look at the equation, we are calculating the energy. We are making an average (integral from -inf to +inf) of the noise squared (we multiply by the p(x) to make an average on ensembles). Saying that we are computing the energy is not correct because it is not the energy of the noise, but the 'energy' of the noise at a particular value. Only if the noise is stationary it holds for all the times, otherwise we need to compute this integral for every time.

It is not energy but power because if we square it we will have an infinite value.



- for **every** instant t_1 the **mean square value** (or variance) $\overline{x^2(t_1)} = \sigma_x{}^2(t_1)$
For stationary noise $\overline{x^2}$ does NOT depend on time t_1
- for **every couple** t_1 and $t_2 = t_1 + \tau$ the **meanproduct** $\overline{x(t_1)x(t_2)} = \overline{x(t_1)x(t_1+\tau)}$
For stationary noise it depends only on the time interval τ, NOT on the time position t_1

We can compute the variance of the noise for each particular time instant. If the noise is stationary, the variance is a number and doesn't depend on time, it is the same for all the times.

We are not interested in the covariance, but something different. With the signal we computed the autocorrelation, now we compute the mean product, in the ensemble domain, not in the time domain. We are making an average on ensembles the product of x(t1) and x(t2). Also in this case, if the noise is stationary, the product depends only on tau and not t.

AUTOCORRELATION OF NOISE

Autocorrelation is not exactly the same as for the signal, it is just a matter of name. Now we are not making an average on time, but we are making the multiplication between two different values on different ensembles. It is autocorrelation Rxx(t1, t1 + tau), which are the two variables.

It is a function of t1 for nonstationary noise. If the noise is stationary, we can write Rxx(tau).

The autocorrelation of the noise is, again, an ensemble value, while the signal autocorrelation is a time average.
Furthermore, if we compute the autocorrelation at zero tau value, we get the mean square value. It is the same connection we had for the signal. One variable still remains and it is t, if the noise is not stationary. If the noise is stationary, the autocorrelation in zero gives us the information about the power.

POWER SPECTRUM OF NOISE

We want to move in the frequency domain also for the noise.

Noise has power-type waveforms (divergent energy → ∞)

which have statistical variations from waveform to waveform of the ensemble.

By **averaging over the ensemble** of the autocorrelations of the noise waveforms ,

the concepts of power and power spectrum **introduced** **<u>for the signals</u>**

can be **extended to the noise**

$$P = \overline{\lim_{T\to\infty} \int_{-T}^{T} \frac{x^2(\alpha)}{2T} d\alpha} = \overline{\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{x_T{}^2(\alpha)}{2T} d\alpha} = \overline{\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{|X_T(f)|^2}{2T} df} =$$

$$= \int_{-\infty}^{\infty} \overline{\lim_{T\to\infty} \frac{|X_T(f)|^2}{2T}} df = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\overline{|X_T(f)|^2}}{2T} df$$

Therefore, the Power Spectrum of the noise is defined as

$$S_x(f) = \lim_{T\to\infty} \frac{\overline{|X_T(f)|^2}}{2T}$$ x

and the noise power is

$$P = \int_{-\infty}^{\infty} S_x(f) df$$

for a noise *x* the autocorrelation $R_{xx}(\tau)$ is an <u>ensemble-average</u>,

for a signal *x* the autocorrelation function $K_{xx}(\tau)$ is a <u>time-average</u>

The **noise mean square value** it is the autocorrelation with τ = 0

$$\overline{x^2(t)} = R_{xx}(t,0)$$

for <u>stationary</u> noise it is <u>constant</u> at any *t*

$$\overline{x^2} = R_{xx}(0)$$

We can extend the power concept just making an average on ensembles (already said), and also the autocorrelation is done with an average over the ensembles. So why not trying to make the same thing for the spectrum? For the signal we had a certain definition of the power, why not adding an average on ensembles also for the spectrum from the formula for the signal power spectrum?
We can bring the average on ensembles inside the integral (after Parseval theorem) and we can define the spectrum as x. It is the same definition but with an average on ensembles added.

Has this formula connections with the previous ones? We never defined the power of the noise, but the means square value, the connection between means square value and autocorrelation is the similar I had between power and autocorrelation in signals. Now I'm defining the power, which is the integral of the spectrum (last formula) and it is the same I had with the signal.

We are just making, at this point, an average on ensembles. For signals, the power spectrum is the Fourier transform of the autocorrelation. Let's do the same for noise with also an average on ensembles.
The problem is that the autocorrelation we considered was Kxx, not Rxx. It is not the autocorrelation we defined for the noise. We would like to merge all together and understand, e.g., the relationship between Rxx and Kxx. If they are different, which is really useful?

By averaging over the ensemble we can extend to the noise

also the second definition of Power Spectrum introduced for the signals

$$S_x(f) = \overline{F[K_{xx}(\tau)]} = F[\,\overline{K_{xx}(\tau)}\,] =$$

$$= F\left[\overline{\lim_{T\to\infty} \frac{\int_{-\infty}^{\infty} x_T(\alpha)x_T(\alpha+\tau)d\alpha}{2T}}\right] =$$

$$= F\left[\overline{\lim_{T\to\infty} \frac{k_{xx,T}(\tau)}{2T}}\right] = \lim_{T\to\infty} \frac{F[\overline{k_{xx,T}(\tau)}]}{2T}$$

The Power Spectrum of the noise can be directly defined as

$$S_x(f) = \lim_{T\to\infty} \frac{\overline{|X_T(f)|^2}}{2T}$$

The noise power is:

$$\boxed{P = \int_{-\infty}^{\infty} S_x(f)df = \overline{K_{xx}(0)}}$$

Autocorrelations similarities

So the power spectrum is the Fourier transform of the autocorrelation Kxx averaged over samples. Kxx is x(t)*x(t+tau) averaged over time (<x(t)x(t+tau)>). The problem is that the directions of time and ensembles are different and uncorrelated. So the average on ensembles can be brought inside the average over time, but if we do so we get Rxx (x).

$$S_x(f) = F[\,\overline{K_{xx}(\tau)}\,]$$

$\overline{K_{x\,x}(\tau)}$ results from the double average,

first over the time $K_{xx}(\tau) =< x(t)x(t+\tau) >$ then over the ensemble

It can be shown that the order of averaging can be exchanged

x $$\overline{K_{x\,x}(\tau)} = \overline{< x(t)x(t+\tau) >} =< \overline{x(t)x(t+\tau)} > = < R_{xx}(t, t+\tau) >$$

The power spectrum thus is related to the ensemble autocorrelation function

$$S_x(f) = F[< R_{xx}(t, t+\tau) >]$$

- For **non-stationary noise** $S_x(f)$ can be defined with reference to the **time-average of the ensemble autocorrelation** function of the noise.

- For **stationary** noise there is no need of time-averaging: it is simply

$$< R_{xx}(t, t+\tau) > = R_{xx}(\tau)$$

and

$$S_x(f) = F[R_{xx}(\tau)]$$

Now we can say that the spectrum is the Fourier transform of the autocorrelation in ensembles averaged over time.
Sx(f) is one power spectrum, it is a function of frequency but it doesn't have the issue of ensembles, because we are making the average over ensembles.
Moreover, for stationary noise, the autocorrelation of the noise, averaged on time, because the noise is indeed stationary, it is exactly the autocorrelation of stationary noise.
Hence we can write the final equation, saying that <u>the spectrum is the Fourier transform of the autocorrelation (if stationary)</u>.

It is an important formula because we have a connection between the power spectrum and the autocorrelation and we can say that the power is the integral of the spectrum, so it is the variance of x because a property of the Fourier transform is that the integral in the frequency domain is equal ot the value in zero in the time domain of the anti-Fourier transform, which is the autocorrelation.
So if the noise is stationary, the power is the mean square value.

BILATERAL AND UNILATERAL SPECTRAL POWER DENSITY

There is a problem, that is the integral is made from -inf to +inf, but we cannot apply this formula. we are making, in fact, an error of factor 2.
When we made the computation of the power, we integrated from -inf to +inf the spectral density. We liked this because we can use the Parseval theorem and move from the frequency domain to the time oan vice versa. The problem is that Sx(f) is symmetrical, so we will consider two times the spectral density.

To solve this issue, the spectral density used so far is called bilateral spectral density Sxb(f), and we call two times the bilateral density and the unilateral spectral density.

- The mathematical spectral density $S_x(f)$ defined over $-\infty < f < \infty$,

  is a **bilateral** spectral density $S_{xB}(f)$

  attention is called on this fact by the second subscript B

- The noise power computed **with the bilateral density** S_{xB} is

$$P = \int_{-\infty}^{\infty} S_{xB}(f)df$$

- Since $S_{xB}(f)$ is symmetrical $S_{xB}(-f) = S_{xB}(+f)$, it is

$$P = 2\int_{0}^{\infty} S_{xB}(f)df = \int_{0}^{\infty} 2S_{xB}(f)df$$

- **A unilateral «physical» spectral density $S_{xU}(f) = 2S_{xB}(f)$ is usually employed** in engineering tasks for making computations only in the positive frequency range

- The noise power computed with with the unilateral density S_{xU} is

$$P = \int_{0}^{\infty} S_{xU}(f)df$$

NOISE SOURCES

NOISE IN DIODES

Real case: Shot current in a diode

- **Random** sequence of many **independent** pulses, i.e. «shots» due to single electrons that swiftly cross the junction depletion layer
- Pulses have rate p, charge q and very short duration T_h (shorter than transition times in the circuits)
- «Shot» current has mean value $\boxed{I = p \cdot q}$
- Shot current has fast fluctuations around the mean, called **shot noise** (or Schottky noise, after the name of the scientist who explained it)
The technical literature reports that shot noise has constant spectral density

$$\boxed{S_{nu} = 2qI} \quad \textbf{unilateral density} \text{ in } 0 < f < \infty$$

A carrier travelling in a junction generates noise. P is the probability that a carrier crosses the junction and q is the charge of the carrier. In this case the average value of the current is not 0, so we have noise associated to a current that is not 0.

Shot noise is associated to any current in the diode, including the signal current and background current. Moreover, the spectral density we compute is the unilateral one.
Shot noise is not exactly 2qI.

Diode noise in forward bias

Let's start from the formula of the current in the diode. We have to pay attention when we are at zero bias; at zero bias we have the exponential term is 0, so we have Is – Is and the current is 0. At this point the noise is not zero, because the 'zero current' is positive current plus negative current, and in noise we sum the square → we have two times the noise in the normal case (4qI).

$$I = I_S\left(e^{\frac{qV}{kT}} - 1\right) = I_S e^{\frac{qV}{kT}} - I_S$$

The diode current is the result of opposite shot components with mean values:

a) $-I_S$ reverse current of minority carriers, which fall down the potential barrier

b) $I_S\, e^{\frac{qV}{kT}}$ forward current of majority carriers,which jump over the potential barrier

- The **mean** current is the **difference** of the components
- The independent current **fluctuations are quadratically added** in the spectrum

$$S_{nU}(f) = 2qI_S e^{\frac{qV}{kT}} + 2qI_S = 2q(I + I_S) + 2qI_S = 2qI + 4qI_S$$

- In forward bias it is $I \gg I_S$ and the spectrum is

$$\boxed{S_{nU}(f) \approx 2qI}$$

- At zero bias it is I=0 and the spectrum is

$$\boxed{S_{nU}(f) \approx 4qI_S}$$

EXAMPLE

The zero bias condition has to be studied for low voltages applications in IC.

NOISE IN RESISTORS

- **The voltage V between the terminals of a conductor with resistance R shows random fluctuations that do not depend on the current *I***
- The technical literature reports that this noise has voltage spectral density S_{vU} constant up to very high frequency >> 1GHz: denoting by R the resistance and by T the absolute temperature it is
  $$S_{vU}(f) = 2kTR \quad \textbf{(bilateral)}$$
- This noise can be described also in terms of current in the conductor terminals: denoting by $G = 1/R$ the conductance, the current spectral density is
  $$S_{iU}(f) = 2kTG \quad \textbf{(bilateral)}$$
- This noise is known as ***Johnson-Nyquist noise***, after the name of the scientists that first studied and explained it.
- It is generated by the agitation of the charge carriers (electrons) in the conductor in thermal equilibrium at temperature T

We are using the bilateral power spectral density. The resistor noise is 4nv/sqrt(Hz) for 1k resistance. Same reasoning can be applied for the velocity of light, which is 30 cm/ns.

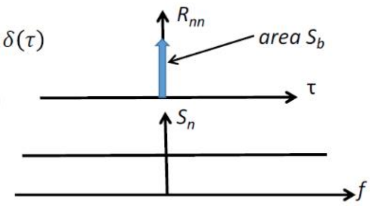
WHITE NOISE

IDEAL «white» noise
is a concept extrapolated from Johnson noise and shot noise
defined by its **essential** feature:
no autocorrelation at any time distance τ, no matter how small

$$R_{nn}(\tau) = S_b \cdot \delta(\tau)$$

and therefore constant spectrum

$$S_n(f) = S_b$$



In reality such a noise does not exist: it would have divergent power $\overline{n^2} \rightarrow \infty$

REAL «white» noise has

- **Very small width** of autocorrelation, shorter than the minimum time interval of interest in the actual case and therefore approximated to zero
- **Very wide band** with constant spectral density S_b , wider than the maximum frequency of interest in the actual case and therefore approximated to infinite

A **white noise is a noise whose autocorrelation is a delta**. The problem that it has infinite power, because in a stationary white noise the value in zero of the autocorrelation is the power and it is infinite. In fact, if we make the Fourier transform of a delta we get a constant, so the spectrum is flat and hence the white noise has infinite power.

So white noise is something that doesn't exists in real world, the white noise will always be limited by someone else.
So we need to approximate the white noise. To define if a <u>noise is white or not, we can say that if the autocorrelation function is narrow with respect to the duration of the signal, that is equivalent to say if the signal is constant over my bandwidth of interest</u>.

Moreover, we might also have non stationary noise, and non stationary white noise. Since it is white the autocorrelation is a delta, and non stationary means that it changes time by time, and the thing that changes of the delta is the area. Also in this case it doesn't exist.

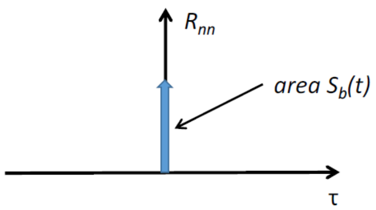
Also in non-stationary cases the IDEAL «white» noise
is defined by the **essential** characteristic feature:

no correlation at any finite time distance τ, no matter how small,
but the noise intensity is no more constant, it varies with time t

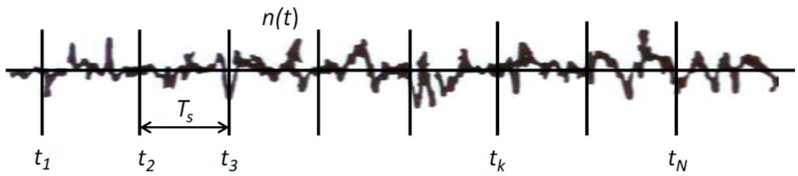
that is

the autocorrelation function is δ-like,
but has **time-dependent area $S_b(t)$**

$$R_{nn}(t, t+\tau) = S_b(t) \cdot \delta(\tau)$$



Filtering white noise



For clarity, let's consider a discrete case:

linear filtering in digital signal processing:

- Sample n_1 at t_1 and multiply by a weight w_1 ,
- sample n_2 at $t_2 = t_1 + T_s$ and multiply by a weight w_2 and sum
- and so on ....

The filtered noise n_f is

$$n_f = w_1 n_1 + w_2 n_2 + \; ..... = \sum_{k=1}^{N} w_k n_k$$

and its mean square value is

$$\overline{n_f{}^2} = \overline{w_1{}^2 n_1{}^2 + w_2{}^2 n_2{}^2 + \; ... + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1{}^2 \overline{n_1{}^2} + w_2{}^2 \overline{n_2{}^2} + \; ... \; w_1 \, w_2 \overline{n_1 n_2} + w_1 w_3 \overline{n_1 n_3} + \cdots$$

What I can do is changing the weight of my samples depending on what I want. I'm making a linear superposition changing the weight of each information. For the noise we need to square and make the average on ensembles (we have also the cross products).

If the noise is white, the autocorrelation is a delta, which has an infinite value at t = 0 ad 0 for any other time. So if the distance between two samples is different from zero, the two samples are uncorrelated. So the average of the cross product will be zero.

Moreover, if the noise is stationary, so the mean square value is the same for any point, we can make the sum. If the noise is not white doing this is more complicated.

$$\overline{n_f^2} = \overline{w_1^2 n_1^2 + w_2^2 n_2^2 + \ \dots + w_1 n_1 \cdot w_2 n_2 + w_1 n_1 \cdot w_3 n_3 + \cdots} =$$

$$= w_1^2\overline{n_1^2} + w_2^2\overline{n_2^2} + \ \dots \ w_1 \, w_2\overline{n_1 n_2} + w_1 w_3\overline{n_1 n_3} + \dots$$

If noise at interval T_s is **not correlated**, then all rectangular terms vanish

$$\overline{n_1 n_2} = \overline{n_1 n_2} = \cdots = 0$$

and the result is **simply a sum of squares**, even in case of non-stationary noise

$$\overline{n_f^2} = w_1^2\overline{n_1^2} + w_2^2\overline{n_2^2} + \ \dots =$$

$$= \sum_{k=1}^{N} w_k^2\overline{n_k^2}$$

If the noise is stationary

$$\overline{n_1^2} = \overline{n_2^2} = \overline{n_3^2} = \cdots = \overline{n^2}$$

there is a further simplification

$$\overline{n_f^2} = \overline{n^2}\,(w_1^2 + w_2^2 + \ \dots) =$$

$$= \overline{n^2}\ \sum_{k=1}^{N} w_k^2$$

we will see later that also with continuous filtering white noise brings similar simplification

Band-limited white noise

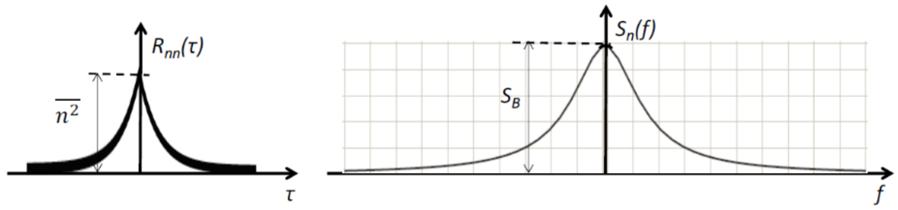
Every time we will have something that limits our bandwidth, the amplifier, the PCB and so on.

Let's consider a single pole. The autocorrelation is the formula x, which is exactly the double exponential. The problem of this noise is nothing in particular, but it is not a delta. However, if tau is very small it seems, but it is not a delta. It is for sure not flat in the frequency domain. So this is not exactly good for the approximation of the white noise.

The real problem is that if we think about the definition of the white noise, the autocorrelation is 0 for any other time than t = 0. The exponential has a non-zero value for any time in the axis. So we would like to approximate the double exponential with something that is easier to be managed, as similar as possible to the white noise from the computations point of view. E.g., to goodly approximate a real white noise, we would like to have a rect in the frequency domain.

- **Real white noise** = white noise with band limited at high frequency
- *The limit may be inherent in the noise source or due to low-pass filtering enforced by the circuitry.* **Anyway, in all real cases there is such a limit**
- A frequent typical case is the **Lorentzian** spectrum: band limited by a **simple pole** with time constant T_p , pole frequency $f_p = 1/2\pi T_p$

X $\quad R_{nn}(\tau) = \overline{n^2} e^{-\frac{|\tau|}{T_p}} \qquad\qquad S_n(f) = \dfrac{S_B}{1 + (2\pi f T_p)^2}$



The rect in the time domain cannot be used in the time domain to approximate the delta (for white noise) because the Fourier transform in the frequency domain is a sinh, which goes negative. The Fourier transform of the autocorrelation is the spectrum, and if we integrate the spectrum we get the power. The sinh has negative part, so if I make the integral do I get a negative power? No, so the model is not correct. We could have used a sinh in the time domain, but it is not easy to be managed in the time domain.

Let's consider an example to demonstrate that the model can be wrong. In a capacitor, the energy is E = ½*C*Vdd^2. If I have a capacitor with no voltage across it, the energy is 0. So the overall energy of the system is E.
This energy 'inside the box' cannot change. Now I connect the two capacitances inside the box; each capacitor will have Vo/2. The new energy would be ½*(Vo/2)*2C. So in theory the energy is decreased. The wrong thing is the model. In fact, the missing energy is dissipated in the resistance of the wire connecting the two capacitors.

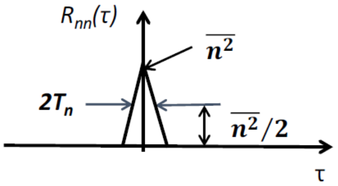
The same for instance applies for LASER light modulation with a sinus. We need to have an offset, otherwise we are like creating negative light.

Simplified description of wide-band noise

We decided that the triangle is a good approximation for the time domain (goes to 0 after a certain area), and for the frequency domain we choose the rect.

The true $R_{nn}(\tau)$ and $S_n(f)$ can be **approximated retaining the noise main features**:

a) equal **mean square $\overline{n^2}$** and b) equal spectral **density S_b**



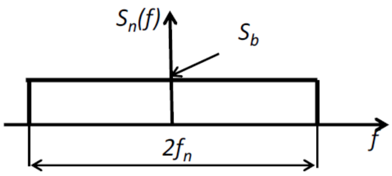
in time: $R_{nn}(\tau)$ triangular approx, half-width $2T_n$

a) equal msq noise : $R_{nn}(0) = \overline{n^2}$

b) equal spectral density: [area of $R_{nn}(\tau)$] = S_b ,

(i.e. $\overline{n^2}\, 2\, T_n = S_b$)

Correlation width = $\Delta\tau = 2T_n$



in frequency: $S_n(f)$ rectang approx, half-width f_n

a) equal msq noise : [area of $S_n(f)$] = $\overline{n^2}$

(i.e. $S_b 2\, f_n = \overline{n^2}$)

b) equal spectral density: $S_n(0) = S_b$

Noise bandwidth: $\Delta f = 2f_n$

Note that $\Delta\tau \cdot \Delta f = 1$ which is consistent with $S_n(f) = F[R_{nn}(\tau)]$

If I approximate the double exponential with a rect I would like to get the same results. To have the same results, I would like to have the same equal msq noise for the rect spectrum. I have to variables: value in t = 0 and area. The second condition that I can apply is that the value in zero of the spectrum (in the frequency domain) is the area in the time domain. So the value in zero in the frequency domain is the area of the autocorrelation → I can say that at least in 0 I want the same power spectrum of the Lorentzian spectrum, which has a shape that is not a rect.

In the time domain again we want the same msq value. As for the width, I know that the area of the autocorrelation is the value in zero of the spectrum and so I set the product of value in zero and 2Tn equal to that.

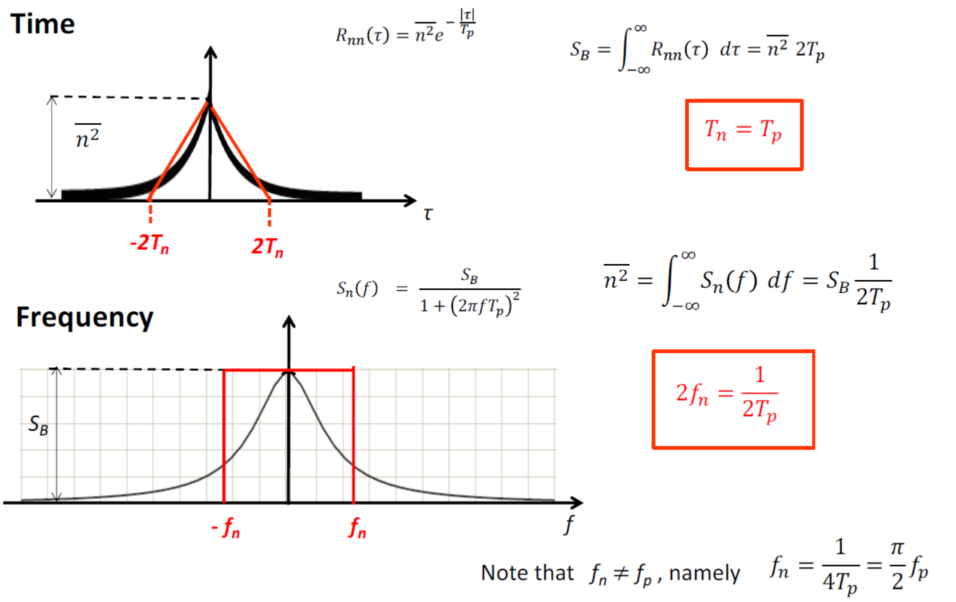
Hence I defined a model both in the time domain and in the frequency domain. This model is working, but there is no link between the model in the time domain and the model in the frequency domain through the Fourier transform.

The idea is that someone has given us the spectrum of the white noise, which is a Lorentzian one, and we want to approximate the real white noise in both the time and frequency domains. In the frequency domain we want something flat up to a certain frequency, and the two approximations are the ones in the previous image.
When we make a model we want a model that is correct and easy to use. We can use a model with a different Fourier transform if we use it just in the time domain, we don't need correlation with the frequency domain, but the result we get must be the same than in the case of the real noise.

The total power and, in the frequency domain, the flat value at 0 Hz are the important parameters for the white noise. So we have one parameter for the frequency domain and one for the time domain.

We know that the value in 0 in the frequency domain, that we want to maintain, is the integral from -int to +inf in the time domain. Similarly, the power is the integral from -inf to + inf of the spectral density.

In the next image we have the superposition of the Lorentzian curves and the approximations.



The bandwidth of the white noise that approximates our Lorentzian spectrum is fn = 1/4Tp.

Power: P = Sv*B, where B is the bandwidth. Sv = 10nV/sqrt(Hz), the amplifier has a BW 500 MHz. To calculate the real noise, B = 2 * 1/4Tp, where Tp is connected to the pole of the amplifier (2 because bilateral).

The real white noise has zero correlation for any time, but the real white noise has some correlation. However, the Lorentzian shape in the time domain tells us that we have correlation at every time, and this is not good. I'd like to have something that doesn't involve and exponential decay time in the time domain, or similarly, that is as flat as possible in the frequency domain.

Sb is the area of the autocorrelation, which is has to be equal to the area of the triangle, because the area of the autocorrelation is the value in 0 of the spectral density, that I want to maintain. In the frequency domain we would like a rect instead of the Lorentzian spectrum; the value in zero is the same, and then the area of the approximation and of the real spectrum has to be the same. The area of the real spectrum is the power (n^2_bar), integral from -int to +inf of the Sn(f).
So <u>to approximate the Lorentzian spectrum with a rect we need to choose the same value in zero and a fn value that gives us the same power</u>.

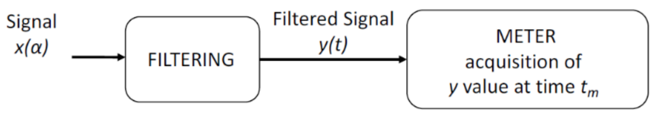
So for instance n^2_bar = Sb*2fn, where Sb is the bilateral spectral density (= Suni*fn).

FILTERING SIGNALS

- Discrete-Time and Continuous-Time Signal Filtering
- Filter Weighting Function in the Time Domain
- Linear Filters with Constant-Parameters
- Time-Variant Linear Filters
- Weighting Function in the Frequency-Domain

The filtering of the signal is very easy to be done, we need to create the weighting function.
The goal is to choose a particular time t_m in the time axis and I want to compute the value of the signal at t_m. How do I choose t_m? So far we won't consider this problem. What if we want the filtered value of the signal at t_m? I can use the information of the signal that I have before t_m to improve the value of the signal at t_m.
The past events can be used to improve the SNR. This is done with a linear system, which is a linear superposition of different events properly weighted → we make a linear superposition of past events.

Digital filter approach



- **Linear filtering** = the superposition of effects is valid
- The output is a weighted sum of input values x taken at various times α with **weights that do NOT depend on the input x**

In **discrete-time** filtering (*e.g. in digital signal processing DSP*)

$$y(t_m) = w_1 x(\alpha_1) + w_2 x(\alpha_2) + w_3 x(\alpha_3) + \cdots + w_n x(\alpha_n) = \sum_{k=0}^{n} w_k x(\alpha_k)$$

$$y(t_m) = \sum_{k=0}^{n} w_k x_k$$

If the weights are the **same set w_k for any t_m** (for any acquisition time) it is a **constant-parameter** filtering ; otherwise, it is a time-variant filtering

I have a lot of numbers from the past and I want them to create the new value. I can sum the samples with proper weights. The weight must not depend on the input signal (if a filter has the weights depending on the signal we are considering adaptive filtering), and it is not constant.

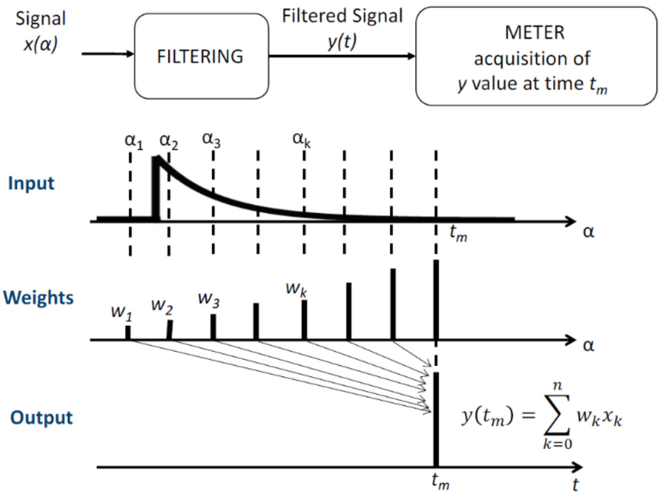
Then we just take the output as sum of each sample times a weight. We can define two types of filters:

- Constant parameter filter: the weight that I use is the same for all the t_m.
- Non Constant parameter filter: the weight that I use is different as a function of t_m.

It is a digital approach because we are using samples.

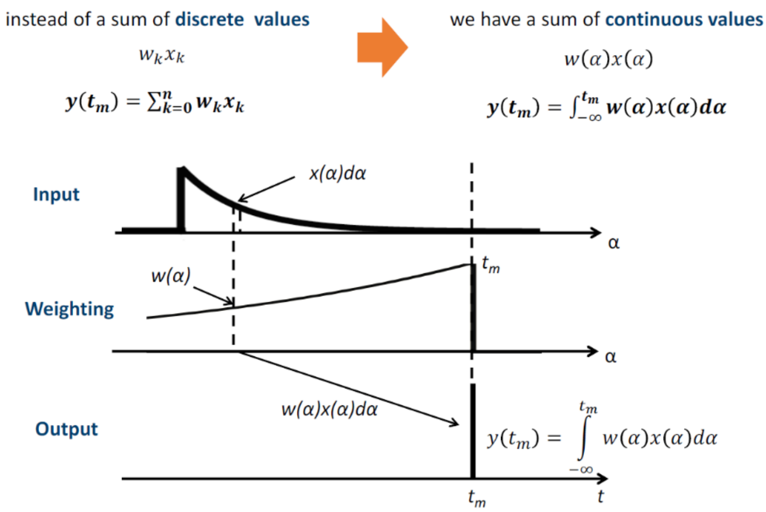
DISCRETE-TIME SIGNAL FILTERING

We want to create a filtered signal creating a filter. The idea is someone is sampling the signal and I have just to define the weight for each sample and then, in order to create the output signal, I create the samples, multiply by the weight and make the sum.
This is for the digital approach.



CONTINUOUS TIME SIGNAL FITLERING

The equivalent of the sum in the digital approach but in the analog one is the integral, while the weights are translated into weights as function of the integration variable.



Just to be clear, the integral might be from -inf to +inf, and it seems strange because +inf is the future. But this is done e.g. because we want to use the Perceval theorem. To extend the integral, we simply consider the weight function as equal to 0 after t_m, so that we can extend the integral to +inf.

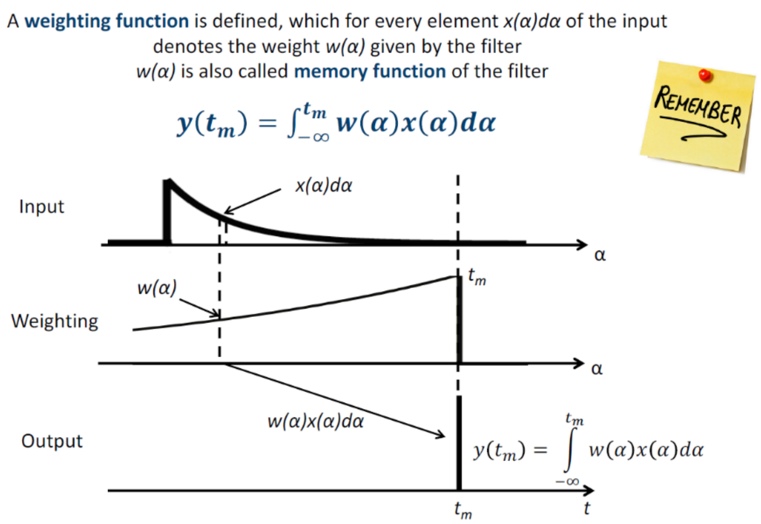
<u>I need to find a way to choose t_m such that the SNR is maximized</u>.

We notice that the weight function is indeed a function, we take the input, multiply by it and make the integral.

WEIGHTING FUNCITON OR MEMORY FUNCTION

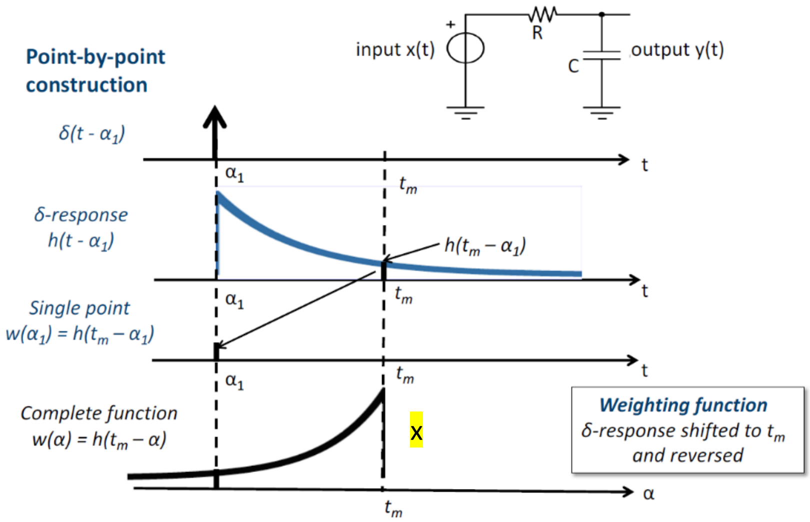
It is a function that gives the weight for every time that I have to apply to the function. It is called also memory function because it gives the amount of memory that I have in the past of the signal.

The idea is the one already introduced before, multiplying and then integral.



CONSTANT PARAMETER LINEAR FILTERS

Linear filter is needed because we use a linear system. Constant parameter means that the weighting function is the same for all the t_m. All the analog filters, e.g. the RC filter, are constant parameters filter.

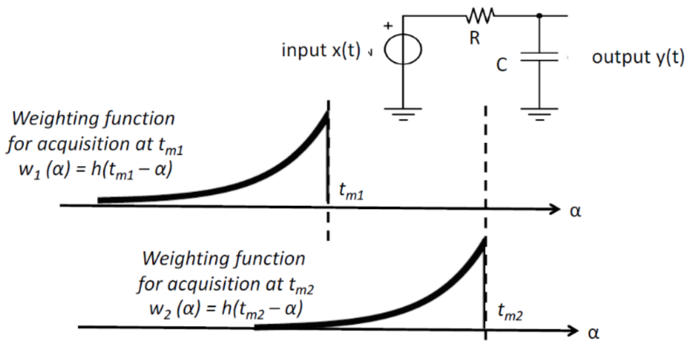


Let's consider a LP RC filter. We will use a graphical approach to define how to create the weighting function.
I take the time axis, I choose (or I'm given) the t_m and I want the value of the weighting function at time alpha_1. I have to apply a delta at time alpha_1 and I look at the delta response at alpha_1 of the LP filter.
Then I take the value of the delta response at t_m and I use it as the value of the weighting function at time alpha_1. This is what is done in the image above.

If I make this calculation for the different times I get the last graph x.

If I change t_m, the weighting function is simply translated, because for a constant parameter filter the weighting function has to be the same (in terms of shape) for all the time instants.

We notice that in this case the weighting function is not so different from the delta response of the LP filter, but it is in the opposite direciton.



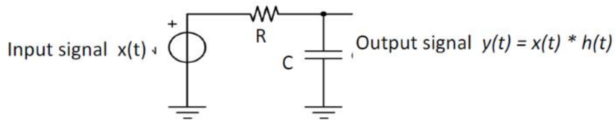
For **constant-parameter** filters the **weighting** function for any t_m has

a) always the same shape and

b) always the same time position **with respect to t_m** .

In other words, when $\mathbf{t_m}$ is changed w(α) changes in a very simple way :
"it walks with t_m just like a tethered dog follows his boss"

We know that the output of any filter is the convolution of the input times the delta response of the filter. It seems, from a graphical point of view, that the weighting function is the delta response, shifted and flipped. Can we confirm this mathematically? The response is in the image.



We have seen that for constant-parameter filters (*but NOT for time-variant filters !*)

$$w(\alpha) = h(t_m - \alpha)$$

This conclusion is confirmed analytically. The weighting function $w(\alpha)$ is definied by

$$y(t_m) = \int_{-\infty}^{t_m} x(\alpha) w(\alpha) d\alpha$$

But for constant-parameter filters (*and NOT for time-variant filters !*)

X $$y(t_m) = x(t) * h(t) = \int_{-\infty}^{t_m} x(\alpha) h(t_m - \alpha) d\alpha$$

The equations above are both valid for any acquisition time t_m , therefore it is

$$\boxed{w(\alpha) \equiv h(t_m - \alpha)}$$

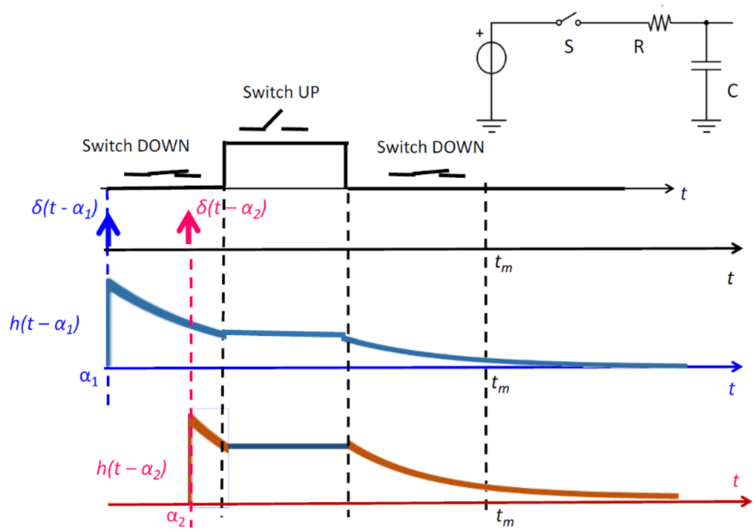
In x the output is the convolution of the input and the delta response and it is actually the definition of the convolution (it should be from -inf to +inf, but we can have the weightening function to 0 after t_m). The result is that the weightening function is indeed the delta response flipped and translated.
A weightening function equal to the delta response would be impossible, because we cannot weight the future.

TIME-VARIANT LINEAR FILTER

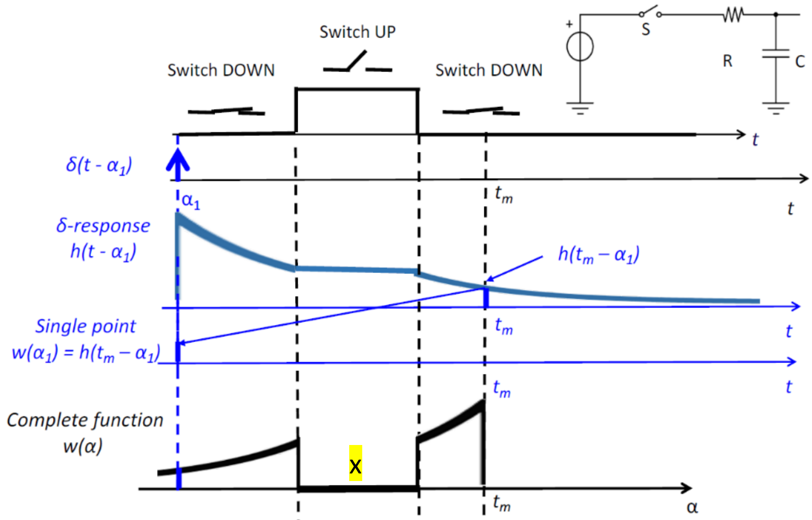
It is a filter that changes as a funciton of the time, not as a function of the input. It is the opposite of the constant parameter filter. One example of this filter is a switch added to a CR.

Let's consider an RC (not CR!) and a switch. I have also to decide when to open and close the switch.
I want to understand the weightening function of this situation.

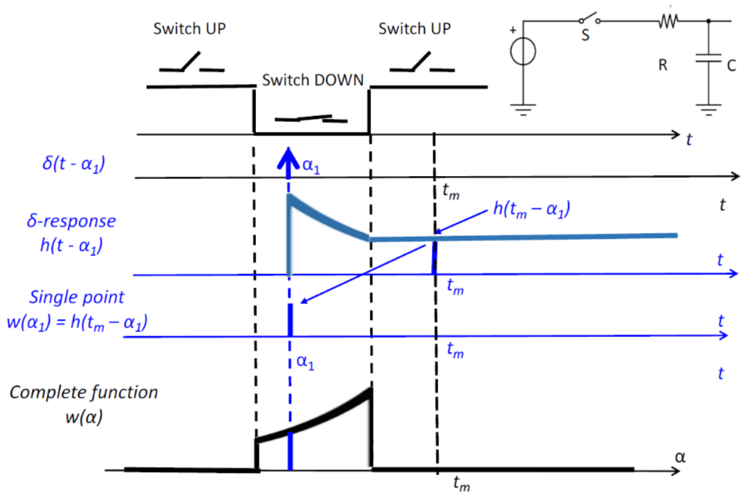
Again, I apply a delta, I study the effect of the delta on the output at the time t_m and I use that value as a value of weighting function for the time where I applied the delta.
I have the switch, I apply a delta, and what happens is that I have a decay time because the switch is closed and then, when the switch is open, there is no current and so the response remains constant.
So I create the delta response and if I apply another delta I will have a different situation.



Now I have to retrieve the value of the weighting function. At t_m the value is 1, but now I don't have to mirror the delta response, it would be wrong. In fact, if I apply a delta when the switch is open, the output is always 0. This is the reason why when the switch is open the weighting function is 0 (x).



Let's now do the opposite. The approach is exactly the same. So I apply a delta, i look at the output and use that vale for the weighting function.



WEIGHTING FUNCTION IN THE FREQUENCY DOMAIN

The concept of acquired value y(t) as a weighted sum of components can be extended to the frequency domain. Parseval's theorem

$$\int_{-\infty}^{\infty} a^2(t)dt = \int_{-\infty}^{\infty} A(f)A^*(f)df = \int_{-\infty}^{\infty} A(f)A(-f)df$$

can be extended to the product of two functions *a(t)* and *b(t)*

$$\int_{-\infty}^{\infty} a(t)b(t)dt = \int_{-\infty}^{\infty} A(f)B^*(f)df = \int_{-\infty}^{\infty} A(f)B(-f)df$$

Denoting by $W(f) = F[w(t)]$ we have

$$\boldsymbol{y(t_m) = \int_{-\infty}^{\infty} x(\alpha) \cdot w(\alpha)d\alpha = \int_{-\infty}^{\infty} X(f) \cdot W(-f)df}$$

The value y acquired at time tm at the filtering system output can be considered

- either as a weighted sum of instantaneous input values x(t) with weights w(t)
- or as a weighted sum of Fourier components X(f) of the input signal x(t) with weights W(-f) = F[w(-t)]

What done so far was in the time domain, but we want to switch to the frequency domain. For the frequency domain we use Parseval. A*(f) is the conjugate of A(f). The same theorem can be applied if a(t) and b(t) are two different funcitons.

Summary

- For **constant-parameter** filters (and only for them!) the weighting function is simply the **δ-response function reversed and shifted** in time.
- That's NOT true for **time-variant** linear filters, which **do not have a unique δ-response**. The shape of the δ-response depends on when the δ-function is applied to the input during the evolution in time of the filter.
- The weighting function in linear time-variant filters may be difficult to compute, but it always exists.
- For filters that vary in time with simple law it is fairly simple to compute the weighting function, in particular for switched-parameter filters (see previous examples).
- Switched-parameter filters undergo abrupt changes at the transition from a time interval to the next one, but within each interval the parameters stay constant.
- The values of electrical variables (voltages, currents) before and after the switching must be carefully checked because they can be discontinuous, i.e. they may exhibit abrupt variations at the switching time.

In the end, what is t_m? For a constant parameter filter it is very easy to be chosen, because the result of a constant parameter filter is the result of the convolution. To maximize the SNR, the value of t_m to choose is the maximum of the output, because the noise is always the same, and to maximize the SNR I have to maximize the signal.

For non constant parameter filter it is harder. The output of a non constant parameter filter is not the convolution, but it is a number, not a function. To apply a non constant parameter filter we have to firstly change the weighting function, because it changes as a function of t_m. To choose the weighting funciton we have to choose t_m first, then we have to understand the weighting function at that t_m and apply the formula.

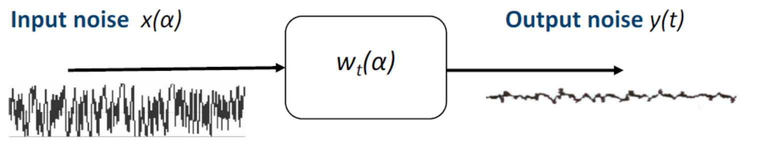
FILTERING NOISE

We need to extract the **power** of the noise, to underststand the type of noise we are dealing with, and the shape of the noise, thanks to **autocorrelation**.

MATHEMATICAL FOUNDATION TO MANAGE NOISE – NOISE FILTERING

Our goal is to derive the autocorrelation of the noise.
I have a noise and a filter, and an output noise. We want to understand the output of the noise at the output of the filter, in terms of autocorrelation and power.

Someone gives us the autocorrelation of the input noise Rxx, and we can write its definition (x). On the right there is what we want to derive, mathematically, that is the autocorrelation of the output Ryy.
In the middle there's the filter.



characterized by $R_{xx}(\alpha, \alpha+\gamma) = \overline{x(\alpha)x(\alpha+\gamma)}$ x

characterized by $R_{yy}(t, t+\tau) = \overline{y(t)y(t+\tau)}$

The **output autocorrelation** can be obtained in terms of the **input autocorrelation** and of the filter **weighting** function :

$$R_{yy}(t_1, t_2) = \overline{y(t_1)y(t_2)} =$$

$$= \overline{\int_{-\infty}^{\infty} x(\alpha)w_1(\alpha)d\alpha \cdot \int_{-\infty}^{\infty} x(\beta)w_2(\beta)d\beta} = \iint_{-\infty}^{\infty} \overline{x(\alpha)x(\beta)} \cdot w_1(\alpha)w_2(\beta)d\alpha\, d\beta =$$

$$= \iint_{-\infty}^{\infty} R_{xx}(\alpha, \beta)\, w_1(\alpha)w_2(\beta)d\alpha\, d\beta$$

We have the data x, but this formula is connected with the weighting function, and we can calculate the output as a function of the input and of the weighting function.
We can write the output autocorrelation at two different times t1 and t2. The first integral is at time t1, then second at t2, and then we take the average of ensembles.

An average on ensembles is something related to the statistics, but w1 and w2 are not statistic variables, so we can bring the average inside the integral and average just the statistic variables x(alpha).

Them x(alpa)x(beta) averaged on ensembles is the autocorrelation of the input. So the autocorrelation of the output is the integral of the autocorrelation of the input times the weighting functions at t1 and at t2. This is the formula that gives us the output noise in any case.

Starting from this formula, we change the name of the variables to prepare it for stationary noises, where the autocorrelation depends only on the difference between two time instants, and not on t1 and t2 independently.

by setting in evidence the intervals of autocorrelation at the input $\gamma = \beta - \alpha$ and at the output $\tau = t_2 - t_1$ can be expressed as

$$\boxed{R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma}$$

and in particular the **mean square noise** at time t_1 is

x $$\boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \iint_{-\infty}^{\infty} R_{xx}(\alpha, \alpha + \gamma)\, w_1(\alpha)\, w_1(\alpha + \gamma) d\alpha\, d\gamma}$$

NB: these equations are **valid for all cases of noise and linear filtering**, that is, also for non-stationary input noise and for time-variant filters.

Once the substitution is performed, we can rewrite Ryy. At this point we need to define the power, that is the mean square value of the noise, i.e. the autocorrelation at tau = 0. We will end up with the expression x, which is the power of the noise only if the noise is stationary. The power of the noise is a number, but here we have a function of t1, so it cannot be power. If the noise is stationary, the mean square value at any time is the same, but if it is not stationary it is different, and to compute the power we need to take an average on time of all the possible mean square values.

This formula holds for any linear filter, not for adapting filter, because to derive the expression x we used the weighting function.

FILTERING STATIONARY NOISE

In a stationary noise, the variance doesn't depend on the time instant, and the autocorrelation depends only on the time distance between two time instants. The noise is the same for every time instant.
So we can say that the autocorrelation of the input is just a function of gamma, time distance.

In case of stationary noise the input autocorrelation depends only on the time interval γ

$$R_{xx}(\alpha, \alpha + \gamma) = R_{xx}(\gamma)$$

The output autocorrelation is correspondingly simplified

$$R_{yy}(t_1, t_1 + \tau) = \iint_{-\infty}^{\infty} R_{xx}(\gamma)\, w_1(\alpha) w_2(\alpha + \gamma) d\alpha\, d\gamma =$$

$$= \int_{-\infty}^{\infty} \left(R_{xx}(\gamma) \left(\int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha + \gamma) d\alpha \right) \right) d\gamma$$

NB: with stationary input noise:

a) a constant parameter filter produces stationary output noise.

b) a time-variant filter can produce a non-stationary output noise!

The autocorrelation of the output still is a function of t1 and tau. It makes sense because the input noise is stationary, not the output one. if the filter is a constant parameters filter, also the output is, but in the case of a non-constant parameter filter the output won't be, because we are changing the filter as a function of the time.

We notice that the inner integral is the autocorrelation function. Going on with the calculations:

$$R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha+\gamma) d\alpha \, d\gamma$$

Denoting by $k_{12w}(\gamma)$ the **cross**correlation of the functions $w_1(\alpha)$ and $w_2(\alpha)$

$$k_{12w}(\gamma) = \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha+\gamma) d\alpha$$

We can write

$$\boxed{R_{yy}(t_1, t_1+\tau) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{12w}(\gamma) \, d\gamma}$$

For the **mean square noise** we must consider the **auto**correlation $k_{11w}(\alpha)$ of $w_1(\alpha)$

$$\text{x} \quad \boxed{\overline{y^2(t_1)} = R_{yy}(t_1, t_1) = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) \, d\gamma}$$

To compute the autocorrelation of the output we take the autocorrelation of the input, the crosscorrelation between two points and we make the integral. Formula x holds for any filter.
k11w is the autocorrelation of the filter.

With **stationary input noise** and for **any linear filter** (i.e. both constant-parameter and time variant filters) the **output noise** mean square value can be computed

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma) \, d\gamma}$$

By the Parseval theorem extension and recalling that:

$$F[k_{11w}(\gamma)] = |W_1(f)|^2$$

the output mean square noise can be computed also in the frequency domain

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2 \, df}$$

The Fourier transform of the autocorrelation, the power spectrum, is the absolute value of the weighting function squared. This is useful because we want to compute the mean square value in the frequency domain, because ewe know already how to compute it in the time domain. Using the Parseval theorem we can get it.

FILTERING WHITE NOISE

It is stationary, so we remove alpha, and to identify that is white we consider the area Sb times the shape, which is a delta. We need to plug this in in the original formula.

Formula x is still in the case if the noise was not stationary.

The fact that White Stationary noise has constant intensity (power)

$$R_{xx}(\alpha, \alpha+\gamma) = R_{xx}(\gamma) = S_b \cdot \delta(\gamma)$$

further simplifies the equation of the output autocorrelation

x $$\boxed{R_{yy}(t_1, t_1+\tau) = S_b \int_{-\infty}^{\infty} w_1(\alpha) w_2(\alpha) d\alpha = \boldsymbol{S_b \cdot k_{12w}\,(0)}}$$

and of the output mean square value

$$\boxed{\overline{y^2(t_1)} = S_b \cdot k_{11w}\,(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha}$$

By Parseval theorem we have also

$$\boxed{\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2\, df}$$

k12_w is the crosscorrelation at 0 time, because we are considering a delta for the white noise, but still there is a distance between time t1 and t2.
Then, the mean square value, if not stationary, at time t1 is like in the formula.

Since then I want to compute the power, I want the autocorrelation with the same time in 0, so we have Sb and the autocorrelation of 11, so the two times are the same (k_11).

Summary

The **output mean square** of a filter that receives stationary noise can be computed

in the **time domain** as

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} R_{xx}(\gamma) \cdot k_{11w}(\gamma)\, d\gamma}$$

in the **frequency domain** as

$$\boxed{\overline{y^2(t_1)} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_1(f)|^2\, df}$$

and in case of **white** noise, i.e. with

$$R_{xx}(\gamma) = S_b \cdot \delta(\gamma) \qquad S_x(f) = S_b$$

$$\boxed{\overline{y^2(t_1)} = S_b \cdot k_{11w}\,(0) = S_b \int_{-\infty}^{\infty} {w_1}^2(\alpha) d\alpha}$$

$$\boxed{\overline{y^2(t_1)} = S_b \int_{-\infty}^{\infty} |W_1(f)|^2\, df}$$

In general, the output mean square value is the integral of the product of the autocorrelation of the input and of the autocorrelation of the filter in the time domain. The only constrain is that the noise is stationary. In the frequency domain we consider the spectrum of the noise and the weighting function squared.

FILTERING NOISE WITH A CONSTANT PARAMETER FILTER

It should give us the shape of the white noise limited by a single pole. In a constant parameter filter, the shape of the weighting function doesn't depend on the time t_m, but the function (not the shape) changes. Moreover, a constant parameter filter is a delta response flipped and shifted.

The constant-parameter filters:

- are completely characterized by the δ-response $h(t)$ in time and by the transfer function $H(f) = F[h(t)]$ in the frequency domain
- have weighting $w_m(\alpha)$ for acquisition at time t_m simply related to the δ-response

$$w_m(\alpha) = h(t - \alpha)$$

- therefore have

$$|W_m(f)|^2 = |H(f)|^2$$

They are **PERMUTABLE**. In a cascade of constant parameter filters, if the order of the various filters in the sequence is changed, the final output does NOT change.

They are **REVERSIBLE**. A constant parameter filter can change the shape of a signal, but it is always possible to find a restoring filter, that is, another constant parameter filter which restores the signal to the original shape.

Hence the squared absolute value of the weighting function is the absolute value squared of the Fourier transform of the delta response.

Furthermore, constant parameter filters are permutable, i.e. we can change the order. If we have e.g. three different independent low pas filters, it doesn't matter the order in which they are placed.
In addition, they are reversible. If we apply this filter, we can always go back, doesn't matter the filter we are using, because we can always design the reverse filter. This is possible because in a constant parameter filter, the zero in the transfer function is just one.

Constant parameter filters with stationary input noise

We start from the general expression for the autocorrelation, then we replace the weighting function at time t1 as the delta response at time t1 flipped and shifted. Then the delta responses are just a function of alpha and beta. The second integral is the convolution between the autocorrelation and the delta response at t2, but since the delta response is the same for any time instant, just h(beta). Then we have again another convolution with h(alpha).

In the end, for stationary noise, the autocorrelation of the output is independent on t1, so it is stationary also the output. h(gamma) * h(-gamma) is the autocorrelation of h, k_hh(gamma). Rxx(gamma) is the autocorrelation of the input. Hence in the frequency domain, the output spectrum is the product of the input spectrum and the absolute value squared of the delta response.

It makes sense because the Fourier transform of the autocorrelation of the noise is the power spectrum. The Fourier transform of the autocorrelation of the delta response is the absolute value squared of the Fourier transform, so everything is coherent. But if I want to compute the power, I need to compute the value in 0 of the autocorrelation.

The output autocorrelation is

$$R_{yy}(t_1,t_2) = \iint_{-\infty}^{\infty} R_{xx}(\alpha,\beta)\, w_1(\alpha) w_2(\beta) d\alpha\, d\beta = \iint_{-\infty}^{\infty} R_{xx}(\alpha,\beta) \cdot h(t_1-\alpha) h(t_2-\beta) d\alpha\, d\beta =$$

$$= \int_{-\infty}^{\infty} \left(h(t_1-\alpha) * \int_{-\infty}^{\infty} R_{xx}(\alpha,\beta) \cdot h(t_2-\beta) d\beta \right) = R_{xx}(\alpha,\beta) * h(\beta) * h(\alpha)$$

and taking into account that:

- the stationary input autocorrelation depends only on the interval $\gamma = \beta - \alpha$
- $d\beta = d\gamma$
- $d\alpha = -\, d\gamma$
- the output autocorrelation is also stationary and depends only on the interval τ

$$\boxed{R_{yy}(\tau) = R_{xx}(\gamma) * h(\gamma) * h(-\gamma) = R_{xx}(\gamma) * k_{hh}(\gamma)}$$

and therefore

$$\boxed{S_y(f) = S_x(f) \cdot |H(f)|^2}$$

So the mean square value, the power of the noise (because of stationary also at the output), is the value in zero of the autocorrelation.
Then I can write the same thing in the frequency domain with the Parseval's theorem.

From the output autocorrelation $R_{yy}(\tau) = R_{xx}(\gamma) * k_{hh}(\gamma)$ we obtain for the **output mean square value:**

$$\boxed{\overline{y^2} = R_{yy}(0) = \int_{-\infty}^{\infty} R_{xx}(\gamma)\, k_{hh}(\gamma) d\gamma}$$

and by Parseval's theorem

$$\boxed{\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) |H(f)|^2 df}$$

In the case of **white** input noise $R_{xx}(\gamma) = S_b \delta(\gamma)$ and therefore

$$\boxed{\overline{y^2} = S_b\, k_{hh}(0)}$$

$$\boxed{\overline{y^2} = S_b \int_{-\infty}^{\infty} |H(f)|^2 df}$$

<u>In a constant parameter filter, with input stationary white noise, the autocorrelation of the output is stationary and it is a delta convoluted with the autocorrelation of the delta response, so it is the autocorrelation of the delta response</u>.

The ideal white noise that passes through a LP filter, so we have a single pole that limits our noise, we have an output that is the Lorentzian autocorrelation. If the tau is very short, so the frequency of the filter is high, we can say that the Lorentzian response is a delta.

If I consider the Lorentzian response, the value in zero will be A^2*tau/2, where A is the amplitude of the response, that in a LP filter is 1/tau → 1/(2*tau).
If I consider the unilateral spectral density instead of the bilateral one (like in the image), I need to further divide by 2: Su * 1/(4*tau) = Su * pi/2 * f_pole.

NB: the autocorrelation function has always the maximum in zero.

LOW PASS FILTERS

SIGNALS AND NOISE

- Signals carry information, but are accompanied by noise
- The noise often is non-negligible and can degrade or even obscure the information
- Filtering is intended to improve the recovering of the information
- Filtering must **exploit** at best the **differences** between signal and noise, taking well into account **what kind of information** is to be recovered. For instance: in case of a pulse-signal, is it just the amplitude or is it the complete waveform?

LOW-PASS FILTERS

- We deal first with «low-pass filters» (LPF), so called because of their action in the frequency domain. The filtering weight is concentrated in a **relatively narrow frequency band** from zero to a limit frequency; above the band-limit it falls to negligible value.
- Correspondingly, in the time domain the weighting function has **relatively wide time-width** (as well as its autocorrelation).
- The action of the filter as seen in time-domain is to produce approximately a **time-average** (i.e. a weighted average) of the input over a finite time interval, delimited by the width of the weighting function

In a lot of cases we manage signals that start from 0 Hz and will finish at a certain frequency, so the signal is centered around zero; this is the reason why LP filters are important.

Moreover, when creating a filter we need to find a way to highlight the difference between the signal and the noise. In some cases we will have a complete overlap between signal and noise and we won't be able to distinguish them. At this point we cannot extract the signal and we will need another approach, the modulation. In the frequency domain, normally the signal is limited in the low frequencies, while the noise is over all the frequency, like in the case of the white noise, or at specific frequencies.

Furthermore, a HP filter can be designed starting from a LP filter (1 – LP filter).

The idea of the LP filter is to **save only a small amount of frequencies** centered around zero. The effect in the time domain is that, since I'm reducing the BW in the frequency domain, **the delta response is larger and larger**. Since I'm trying to reduce the BW, I will have something larger in the time domain, this is the intuitive idea.

Elements to create a LP filter

To understand and to be able to deal with LPF is very important because:

a) LPF are a **basic element** of filtering and a **foundation** for gaining a better insight on all other kinds of filters and better exploit them.

   For instance, a high-pass filter (HPF) can be obtained by subtracting from a given input the output of an LPF that receives the same input. In various real HPF, the physical structure of the HPF actually implements this scheme.

b) LPF are **employed in real cases** of filtering for information recovery

   For instance, in many cases a wide-band noise accompanies signals that have significant frequency components in a relatively narrow frequency band around $f=0$.
   These are not only the cases of DC and slowly varying signals, but also cases where the just the **amplitude** of a **pulse signal** (having fairly long pulse-duration and known pulse shape) must be measured (and not the complete waveform)

We are considering any LP filter, not necessarily the RC one. Other than extracting the useful frequencies, the LP filter is used also because, let's consider e.g. an exponential decay.

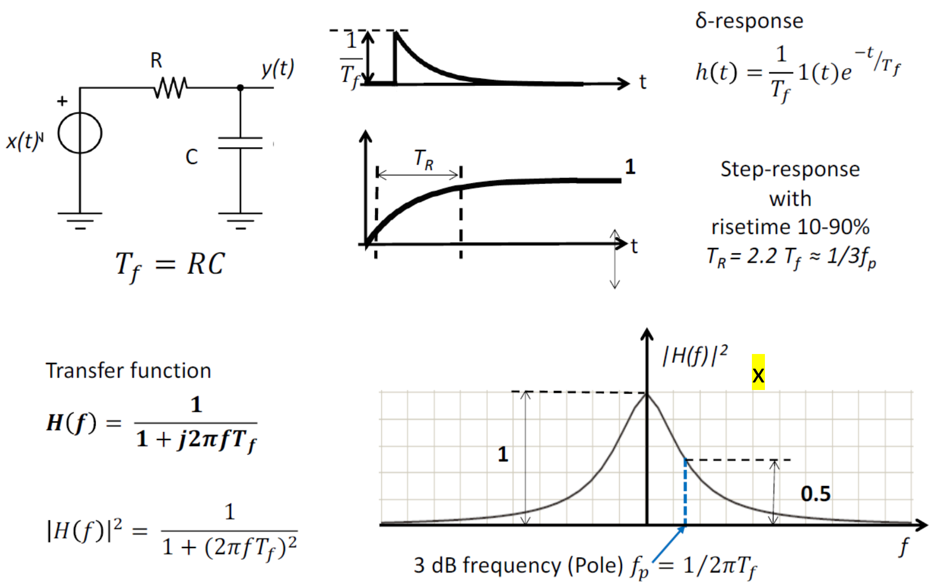
The Fourier transform of 100ns of tau exponential decay time is the LP filter, and the frequency content can be at 1/(2*pi*tau) → signal in the order of MHz.
Instead, if we consider a rect, its Fourier transform is a sinc, and the frequency content lies the most under the first lobe, and the frequency of the first zero to confine the lobe will be 1/T.

So the BW of the signal is in the order of MHz for both the signals in the time domain.

Let's consider an RC as a LP filter, if the BW is 10MHz, the frequency to be chosen is normally one decade after, so that the shape of the signal is not distorted and I'm cutting the noise. Then, depending, the data we want to extract, we need to save either the shape of the signal or the amplitude after the filter. Depending on knowing the shape or not, if known we don't have to preserve it, the important information to be saved is the amplitude.

If we want e.g. to save the shape of the signal, 100MHz would be ok, but to extract the amplitude is enough the BW of the signal. Of course we are not keeping the shape, but we are cutting of one more decade the noise. In this case, the average value of the signal is an important parameter.

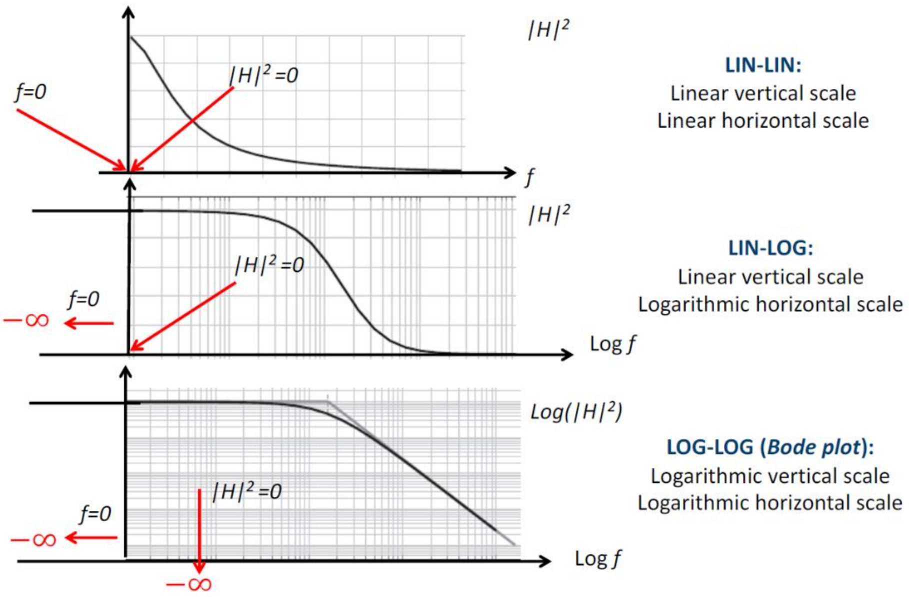
RC INTEGRATOR



We know the delta response of the filter and its amplitude. The other response is the step response; sometimes, in fact, we have the output of the circuit and we don't know how the circuit is implemented, and applying a delta in the real world to the circuit is not easy, whereas applying a step is easy.
To extract the tau, if I have the step response, the time from 10% to 90% is exactly tau, in this way I extract it. It doesn't matter if the step response is not exactly exponential.

Then we know the Fourier transform of the delta response, with plot x that is the one in the linear-linear axes. The point where we have 0.5 of the peak value, in linear scale, is the pole frequency.

Frequency response



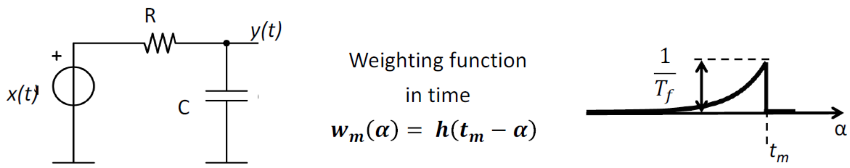
The LIN-LOG plot is useful for the 1/f noise. All these 3 are 3 different plots of the same response.

The power of the noise is the integral from -inf to + inf of the spectral density of the noise times the absolute value squared of the Fourier transform of the weighting function of the filter.

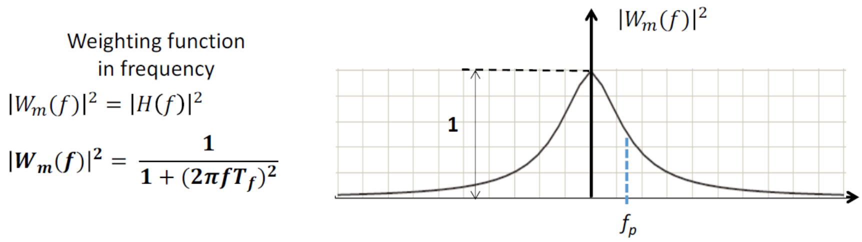
The area under the LIN-LIN response is, by definition, the power of the noise. When comparing filters, we can identify the best one by comparing the area. The one with the **smaller area has the smaller noise**.

We need also LIN-LOG because the previous consideration works with the white noise because the spectral density is flat, but for the 1/f noise it is no more correct to bring out Sb of the integral, because it is no more constant, so the first plot area cannot be used. Instead, the power is the area of the graph for 1/f noise if we use the linear-log scale.

CONSTANT PARAMETER LP FILTER



Output: can be seen as an **average over a time interval** ≈ $2T_f$ preceding t_m



Output: can be seen as a **selection of the lower frequency components** up to ≈ f_p

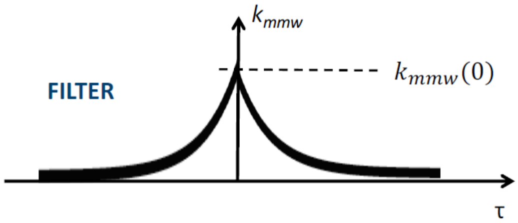
The weighting function is the delta response shifted and flipped. Then the weighting function is equal to an average over a time interval 2*tau (Tf = tau).
An average is take different values, sum them and divide by the number of them. In this case we are considering a weighted average.

Again, we know the weighting function in the frequency domain and plot it, it is again the delta response.

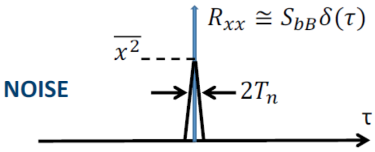
Autocorrelation
In the time domain we know the autocorrelation of the noise, and we need the autocorrelation of the delta response to know the autocorrelation of the output. This is the only information we need in the time domain, the autocorrelation of the filter.
As for the noise, the autocorrelation of the noise is a delta. The autocorrelation of the noise is taken, multiplied by the one of the filter, take the integral and we get the noise power.

NB: something times a delta is the value in zero of the other function.

If the Tn is much smaller than the filter, I can easily use a delta, so using the delta or the approximation is the same. If Tn is larger or comparable with the filter, I cannot use a delta.



$$k_{mmw}(\tau) = \frac{1}{2T_f}\, e^{-\left|\frac{\tau}{T_f}\right|}$$



$$\overline{y^2} = \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot k_{mmw}(\tau)\, d\tau \quad \text{x}$$

The noise is considered wide-band if it has autocorrelation much narrower than the filter weight autocorrelation, that is, if $T_n << T_f$

We can then approximate $R_{xx} \cong S_{bB}\delta(\tau)$ and obtain

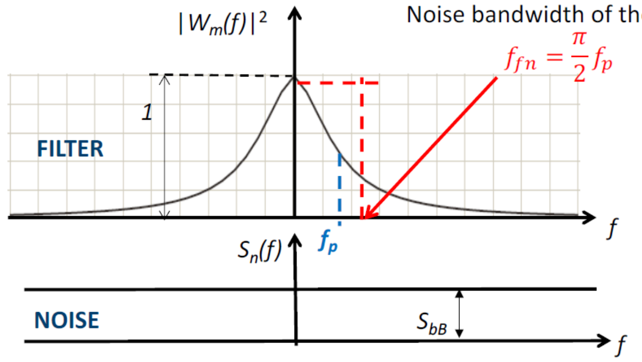
$$\boldsymbol{\overline{y^2} = S_{bB} \cdot k_{mmw}(0) = \frac{S_{bB}}{2T_f}} \quad \text{y}$$

k_mmw is the autocorrelation of the filter.
In the end if I consider the Rxx as a delta, I get the formula found before.

X formula is very important to be remembered. It is crucial in the time domain, and we need the i/o autocorrelation, that typically is a data, and the autocorrelation of the weighting function, which instead in general has to be calculated (for instance the autocorrelation of the RC filter is a double exponential, i.e. Lorentzian shape).
Formula x also allows us to identify if a noise is white or not. I want I white noise because if so I can replace the autocorrelation with the delta and use a simplified equation. To do so, I compare the width of the autocorrelation of the noise with the width of the autocorrelation of the filter. If small, I can use a delta and use the last formula.
Last expression y works only for the RC filter, while the one SbB*k_mmw(0) for any filter.

Sometimes we are not interested in the shape of the signal, but other information like tha amplitude. In these cases, the LP is even more important. However, the shape can still be used in some ways.

Frequency domain



$$|W_m(f)|^2 = |H(f)|^2 =$$
$$= \frac{1}{1 + (2\pi f T_f)^2}$$

$$\overline{y^2} = \int_{-\infty}^{\infty} S_x(f) \cdot |W_m(f)|^2 \, df \quad x$$

The noise is considered wide-band if it has spectrum much wider than the filter weighting spectrum, that is, if its bandlimit $f_n >> f_p$

We can then approximate $S_x(f) \cong S_{bB}$ and obtain

$$\overline{y^2} = S_{bB} \cdot \int_{-\infty}^{\infty} |W_m(f)|^2 \, df = S_{bB} \cdot k_{mmw}(0) = \frac{S_{bB}}{2T_f} = S_{bB} \cdot 2f_{fn}$$

In the frequency domain we have the integral of the spectral density of the noise times the absolute value of the Fourier transform of the weighting function squared.
If the noise is white, its spectral density is flat and we can take it out of the integral (last formula). Formula x is the one to be remembered.

If the bandwidth of the noise is much larger than the one of the filter, we can say that the spectrum of the noise is like the spectrum of the white noise, so its flat. In the end, with the last formula, we get the same result obtained starting from the time domain.

f_fn is the equivalent noise BW. We are defining the integral of the filter as a rect area.

Noise BW of the filter

The input must be white noise.

Noise bandwidth f_{fn} of the filter:

defined with reference to a white noise input S_b as the bandwidth value to be employed for computing **simply by a multiplication** the output mean square noise

$$\overline{y^2} = S_{bB} \cdot 2f_{fn}$$

Since it is

$$\overline{y^2} = S_{bB} \cdot \int_{-\infty}^{\infty} |W_m(f)|^2 \, df = S_{bB} \cdot k_{mmw}(0)$$

for any LPF the correct bandwidth limit f_{fn} is

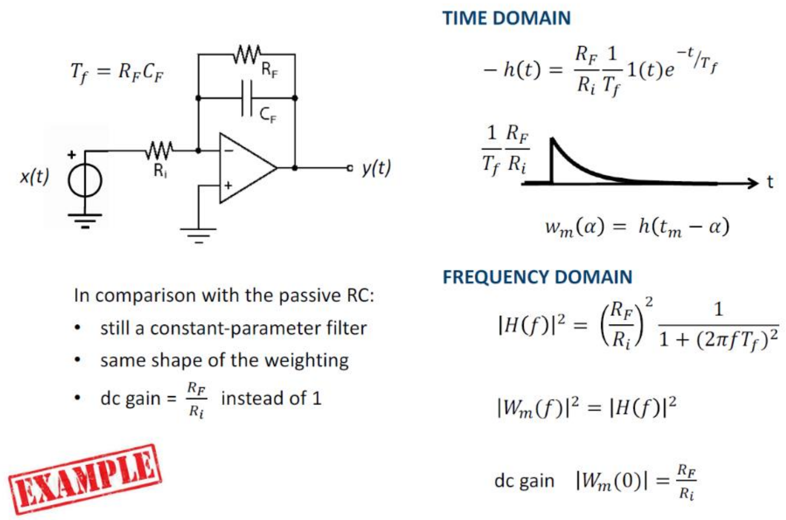
$$f_{fn} = \frac{k_{mmw}(0)}{2}$$

and in particular for the RC integrator

$$f_{fn} = \frac{1}{4T_f} = \frac{\pi}{2} f_p$$

We need to identify f_fn, which is the autocorrelation in 0 divided by 2. This is true for any filter, to the autocorrelation in 0 is a very important parameter. For the RC filter, the autocorrelation in 0 is 1/4Tf.

RC INTEGRATOR ACTIVE FILTER



In the time domain, the delta response is the same of the passive RC, we simply need to change the amplitude both in the time and frequency domains.

This circuit is nothing more than an RC from the signal recovery point of view, because any amplification of a filter doesn't matter.
Our target is the SNR, and every time we have a gain it is the same for the signal and the noise, so the SNR is unchanged.

However, the gain is also important because if we have two stages, the noise of the first stage, since it is multiplied by the gain, dominates the second one, and we can neglect the noise of the second stage. For us it is not important because we have just one stage.

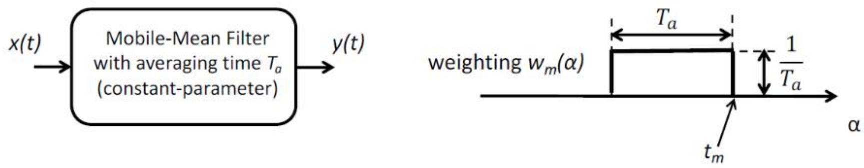
MOBILE MEAN LP FILTER

It is a LP filter where the idea is trying to make an average. We have an input and we want to make an average; we are at t_m, and if we want to plot an average of the signal with the weighting function, the result is a constant for a constant value.
The parameter of the rect weighting functions are width and amplitude. The width Ta defines the time for the average, while the amplitude can be seen in two ways.

Normally, a gain is defined as a ratio at 0 frequency in linear filters (value of the output divided by the value of the input).
We want the value in 0 of the Fourier transform of the rect, but we don't need to know the Fourier transform, because the value in 0 of the Fourier transform is the area in the time domain, which is Ta*1/Ta = 1. In the time domain is input multiplied by weighting function, integrated. So again it's 1.

We are setting the amplitude of the rect to 1/Ta because we are trying to make an average. If the amplitude is changed, no more average filter but some other kind, like an integrator. Hence the one in the image are the rect specifications to make an average.

NB: increasing the width of the filter increases the noise, but we are also increasing of a much higher ratio the signal, so SNR is improving.



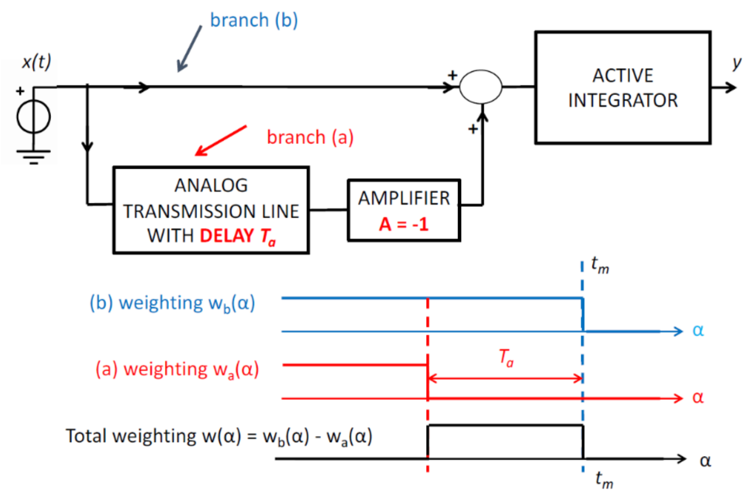
- **A mobile-mean filter (MMF)** produces at any time t_m an output $y(t_m)$ which is not just the integral of the input $x(t)$ over a time interval T_a that precedes t_m, but rather the **mean** value of the input $x(t)$ **over the time interval T_a**, that is, the integral over T_a **divided by T_a**
- In order to obtain this, if we vary the averaging time T_a we must vary inversely the **weight $1/T_a$** (this ensures constant area of $w_m(\alpha)$ i.e. constant DC gain).

The MMF is a **constant-parameter filter**: this is pointed out by the weighting function, which is the same for any readout time tm

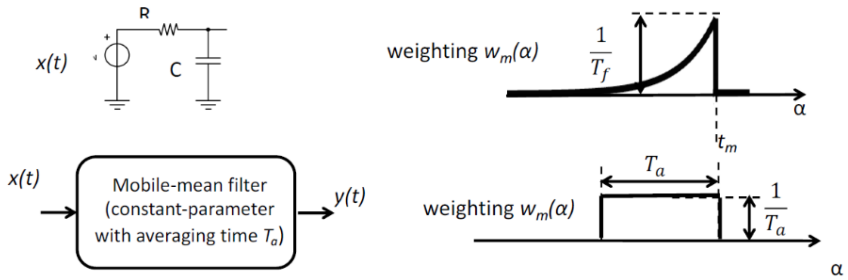
We are still in the field of constant parameter filters, and it is 'mobile mean' because the weighting function is moved in time.

Mobile mean LP filter is a constant parameter filter
Let's consider an active integrator. Its weighting function, for all the time, since I have sum of all the previous time values, is a rect to -inf (blue).
But this is not the weighting function of the MM filter, so I need to add a branch. What I do is to make an integral delayed by a time Ta, and then I have an amplifier with gain -1 (red plot). Then if I sum the two branches I get the weighting function of the MM filter.

The transmission line is nothing more than a cable. The charge speed in the cable is more or less 20 cm/ns.



MM filter vs integrator



The mobile-mean filter produces an output $y(t_m)$ that is exactly the **mean value of the input x over the time interval T_a** preceding t_m.

When T_a is changed, the area of $w_m(\alpha)$ is kept constant, similarly to the case of the RC integrator when T_f is varied (the weight is reduced; the dc gain is kept constant)

Question: can we use a mobile-mean filter as equivalent to a given RC integrator for evaluating the result obtained by processing a signal with low-frequency content in presence of wide-band noise?

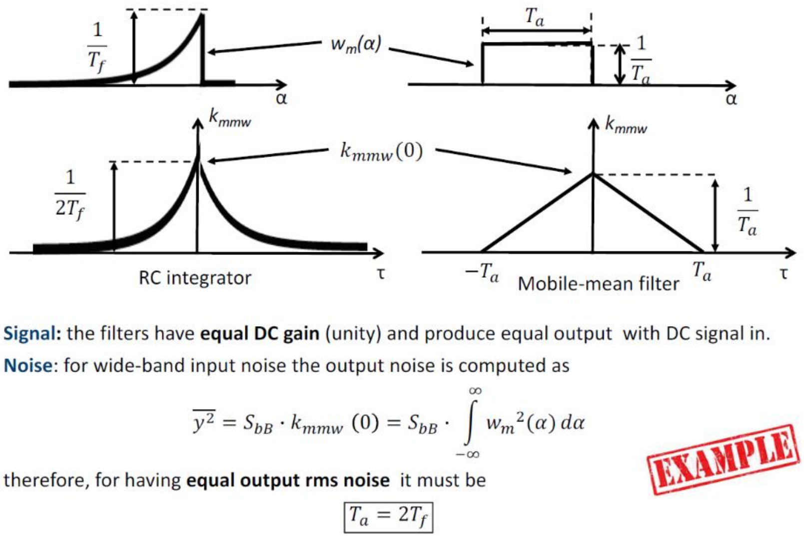
Answer: yes, the time T_a of the mobile-mean filter can be adjusted to produce equal output rms noise of the given RC integrator.

The RC is the equivalent of an average over a couple of tau and the weighting function is not actually an averaging one.
For the MM filter, if the width is increased, the amplitude has to be set to 1/width, and also for the RC the amplitude is 1/tau. How can we compare the two filters?
We look at the SNR giving the same signal in input. If I set the gain equal to 1, the output signal for the two is the same, so **I have just to compare the noise, under the hypothesis that the gain of the two filters is 1**, we can do so.

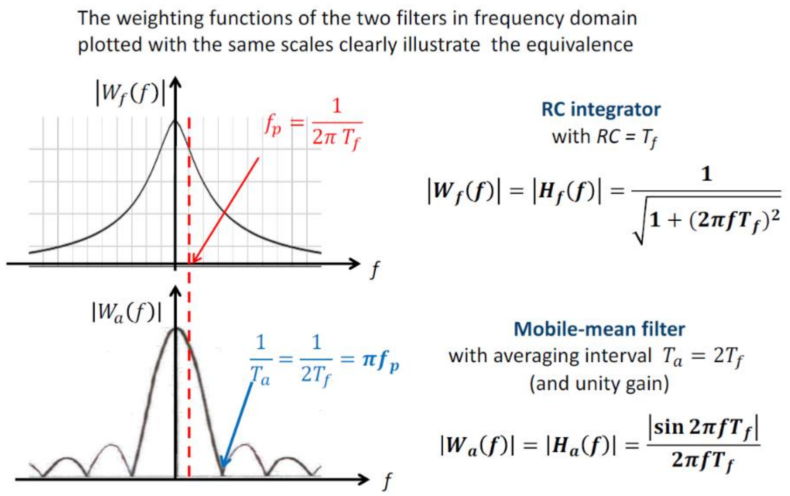
So we need to define the noise of the RC filters. If I have white noise, the noise of the RC filter is the 1/(2Tf), that is the delta times the autocorrelation in 0.
If we make the comparison, we are interested in the value in zero of the autocorrelation. The autocorrelation of the RC is the Lorentzian spectrum (double exponential).

The autocorrelation of the rect is a triangle, and we are not interested in the shape, but in the value in zero. So we take the rect, we multiply it by itself, so 1/Ta^2 of amplitude, and we take the integral over Ta. The final value in zero is 1/Ta.

In the end we need to compare 1/(2Tf) and 1/Ta. If Ta = 2Tf, the SNR of the two filters is the same, and an RC is equivalent to a MM overt two tau of a signal., because it is equivalent to make a MM filter with a width of 2*tau.



Shifting in the frequency domain, on one side we have the Lorentzian spectrum, and the Fourier transform should be a sinc^2, because we need the absolute value squared (in the image just the modulus).



In which case should I prefer the MM over the RC filter? The important thing of the MM filter are the zeros, because at that frequencies the gain of the filter is 0 and we can filter noise at a specific frequency, e.g. at the 50Hz of the power line.

Furthermore, when we considered the rect, its Fourier transform is the sinc, whose zero is 1/width, which practically means that, since the rect is making an average, every time we have an integer number of periods in the width of the rect, we should have a zero in the frequency domain, because the average is 0. So the 0 is when the average is 0, i.e. we have multiple of the periods in the rect width.

BANDWIDTH AND CORRELATION TIME OF LP FILTERS

- The **noise bandwidth** f_{fn} of a low-pass filter is currently employed for evaluating the output noise of low-pass filters in **frequency-domain** computations.

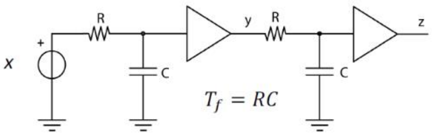
- **A REAL** filter that implements such a «rectangular weighting» in frequency **DOES NOT EXIST**: it would be a non-causal system, with δ-response that begins before the δ-pulse.

- **A REAL** filter that implements such a «rectangular weighting» in time **EXISTS:** it is the mobile-mean filter with averaging time $T_a = T_{fn}$.

- There are, however, practical limitations to the implementation of mobile-mean filters, mainly due to the impractical features and limited performance of the real analog transmission lines with long delay, namely delay longer than a few tens of nanoseconds.

Other constant-parameter LP filters

For LPF filters with real poles, it is often easier to compute the noise bandwidth in time-domain rather than in frequency-domain, because it implies simple integrals (of exponentials and powers of *t*). **Example: cascade of two identical RC cells**



$$h(t) = \frac{t}{T_f^{\,2}} e^{-\frac{t}{T_f}}$$

$$\overline{z^2} = S_{bB} \cdot k_{hh}(0) = S_{bB} \cdot \int_{-\infty}^{\infty} h(t)^2\, dt = S_{bB} \cdot \int_{-\infty}^{\infty} \left(\frac{t}{T_f^{\,2}}\right)^2 e^{-\frac{2t}{T_f}} dt$$

which integrated by parts gives

$$\overline{z^2} = S_{bB} \cdot \frac{1}{4T_f}$$

Since $\overline{z^2} = S_{bB}\, 2 f_n$, the noise bandwidth f_n is

EXAMPLE

$$\boldsymbol{f_n = \frac{1}{8T_f}}$$

SWITCHED PARAMETER FILTERS

Mobile mean filter are limited by the width, we cannot create filters with a width larger than 100ns. With SPF we can create a filter similar to MM, but larger. The output of a SPF is not a function, we are not making the convolution, but it is a number, because with the SPF we choose the t_m and then we compute the signal only at that time, obtaining the result.

Once we understand how to choose t_m with SPF, we will do the same with constant parameter filters.

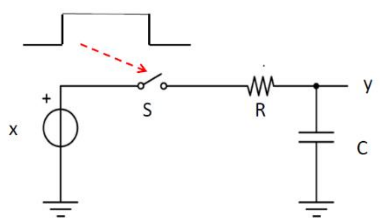
RC LOW PASS FILTER

Let's try to modify the RC to create a non-constant parameter filter. This can be done by adding a switch. Someone has to tell us when the switch is open or closed. when we choose a constant parameter filter we don't have to worry about the switch, in a NCPF (non-constant parameter filter), we need to define when the switch is open and when it's closed.

We need for instance to understand when the signal starts and ends to get when to close the switch. So normally someone is giving us a sync, synchronization signal synchronized with our experiment. At this point I know when to close the switch.
The point is that we don't have always the sync, but sometimes we still create a workaround, in other situations there is no way to know and we cannot use a NCPF.

For instance, if a system responses with an exponential, we perfectly know when the signal starts because we have a peak at the beginning and we know when the laser is given to the sample to get a response (time instant). If instead we send a laser to the satellite, we know when we send the signal, but we don't know when we will receive the signal wrt when we sent it (we don't have the sync, hence), because it depends on the distance. However, in this latter case the NCPF can still be used, and the workaround is based on a repetitive approach.

Instead, there is no workaround if e.g. we are collecting a signal from a star in the sky. We are receiving a signal that comes from the past and we don't have any sync signal for sure.

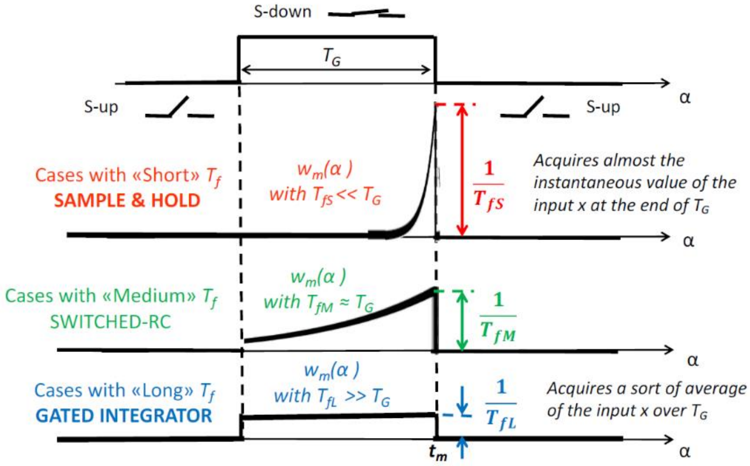


- **State with *S* down (closed in short circuit):** the circuit behaves like a constant-parameter RC integrator; current can flow in and out of C
- **State with *S* up (open circuit):** the circuit is in HOLD, no current can flow, the charge previously stored in C is maintained, the voltage on C stays constant.

In the cases here considered:
(a) the initial state is with *S* open and zero charge in C
(b) the command closes *S* in synchronism with the signal to be acquired
and re-opens S after the acquisition

We have to know when the switch is closed, and when it is open we hold the value stored on the capacitor. Which is the initial state of the switch and of the capacitor? We start from a discharged capacitor and switch open.

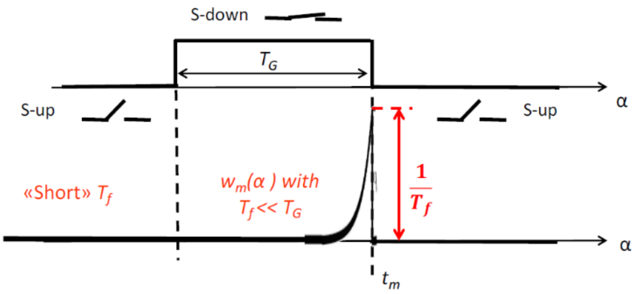
We close the switch and we want to understand the weighting function. If the tau of the RC (Tfs) is much smaller than the Tgate, the discharge finishes before the end of Tgate. The amplitude is of sure 1/Tfs (classical RC response).



It I increase the tau, we get the green result, but we have a part of exponential. The last case is if the tau is very long with respect to the Tgate, and we get a constant value, because the discharge is very long (extreme case of the linear discharge).
The red filter is very useful, not so good from the filtering point, and the blue one is really useful. The green is practically useless.

SAMPLE AND HOLD

The first one is the sample and hold. This red weighting function is a S&H but it isn't a delta. From the math point of view, if we have a signal and this is the weighting function, to get the output of the NCPF we need to integrate from -inf to t_m the product between signal and w_function (weighting function).
If we reduce the tau, the portion of the signal we are acquiring is reducing a lot (the response shortens) → we get a good approximation of a delta. Then as soon as I open the switch I'm saving the value on the capacitor C.



The S&H has **unity DC gain** (C is fully charged at the input voltage within T_G)

$$W_m(0) = \int_0^{\infty} w_m(\alpha) d\alpha = 1$$

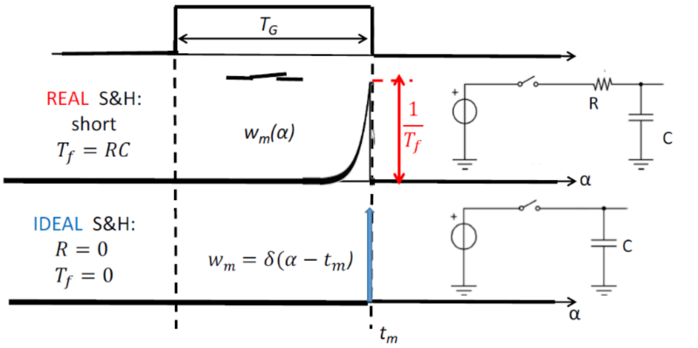
The S&H has very mild filtering action, equivalent to that of a constant-parameter RC integrator with equal time constant T_{fS}. With wide-band input noise S_b (bilateral)

$$\overline{y_n^2} = S_b \cdot \frac{1}{2T_f}$$

The DC gain of the filter should be 1 if I'm creating a S&H, because I want to freeze the input signal value. In DC, the gain in the frequency domain is the value in 0, so the integral from -inf to +inf in the time domain, so the area of the weighting function, so I need an area of the exponential equal to 1.
The area of the exponential with amplitude A and decay time tau is A*tau. So in our case the area is exactly 1.

As for the noise, we have the w_function of an RC, so we know the autocorrelation of the RC in 0, which is 1/2Tf.

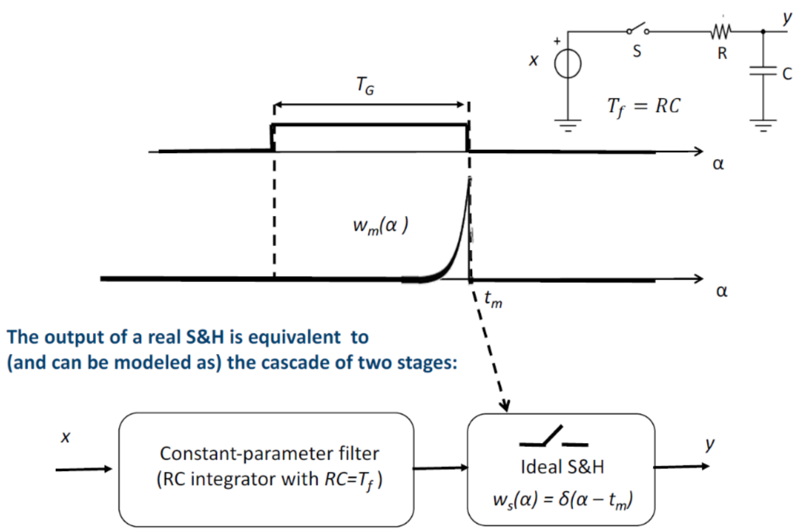
Real and ideal S&H



- The **minimum available T_f is limited by the technology** of devices and circuits (finite R values of fast switching devices and C values required for holding information)
- **S&H acquisition time** = time for reaching the full output value ≈ a few T_f , i.e. currently some tens of nanoseconds in discrete-component circuits some tens of picoseconds in integrated circuits with minimized capacitances

If we reduce R to get closer to an ideal S&H, tau is reduced, so 1/tau increases and we are increasing the noise. We are picking more noise because the noise is spread over all the frequencies, and as soon as we shrink in the tie domain the weighting function, we are spreading it in the frequency domain, so we are increasing the BW in the frequency domain and collecting more noise. So **we can approximate the filter with a delta response, but we are also increasing the noise**.

S&H equivalent model and readout noise



Which is the noise of the S&H?
It is kT/C, so it is independent on R. Using the bilateral noise spectral density, we know that the only source of noise in our circuit is the resistance R. Then we know that the weighting function is the one of the exponential and its autocorrelation (that is the double exponential).

The readout noise is the bilateral spectral density times the autocorrelation in 0, because in input we assume to have a delta in the time domain.

- **READOUT NOISE** of a sampling circuit is the contribution to the output noise due to the internal noise sources in the sampling circuit itself
- In the S&H the main source of readout noise is the wide-band Johnson noise of R with spectral density $S_{bB} = 2kTR$ (bilateral)

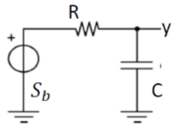
Since

$$w(\alpha) = \frac{1}{T_f} e^{-\frac{(t_m-\alpha)}{T_f}} 1(t_m - \alpha) \quad \text{and} \quad k_{ww}(\tau) = \frac{1}{2T_f} e^{-\frac{|\tau|}{T_f}}$$

the readout noise is

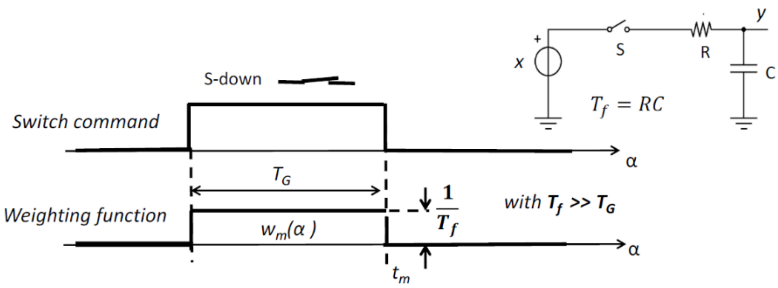
$$\overline{y_R{}^2} = S_{bB} \cdot k_{ww}(0) = 2kTR \cdot \frac{1}{2T_f} = 2kTR \cdot \frac{1}{2RC}$$

$$\boxed{\overline{y_R{}^2} = \frac{kT}{C}}$$



this is just the noise generated and self-filtered by a constant parameter RC filter and is INDEPENDENT OF THE R VALUE, in agreement wth the S&H circuit model.

GATED INTEGRATOR (GI)



- For behaving as GI (uniform weight in T_G) the circuit must have $T_f >> T_G$
- Therefore, the **DC gain G is inherently much less than unity**

$$G = W_m(0) = \int_0^{\infty} w_m(\alpha) d\alpha = \frac{T_G}{T_f} \ll 1$$

- A GI has remarkable filtering action on a wide-band input noise, that is, on noise with autocorrelation width much shorter than the gate duraton T_G .
- Long gate duration T_G is well feasible in practice, much better than a long averaging interval T_a in a mobile-mean filter

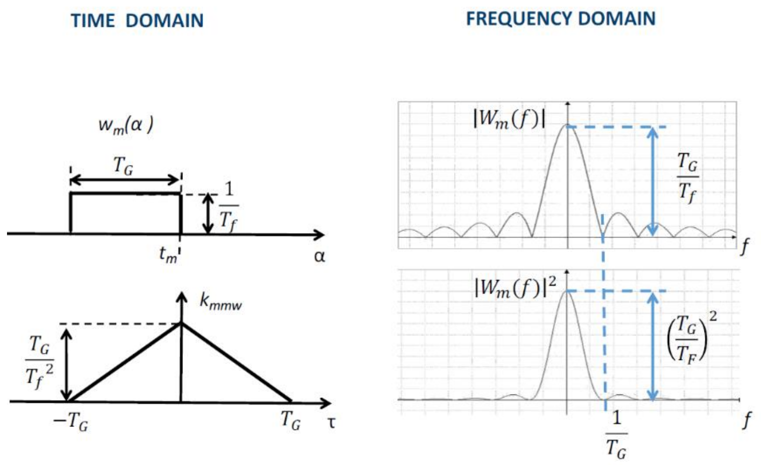
Gated because we choose when we start and end the integration. In this case I'm choosing a Tf, tau, which is much larger than Tg, so I have a rect. Since 1/Tf is very small, the DC gain (value in 0 of the rf frequency response) is much less than the unitary gain (Tg/Tf << 1).

Furthermore, since the gain is not 1, it is different from the other filters, so when we compare filters we cannot just compare the noise, but we have to compare the SNRs.
A solution would be to use an active integrator to recover the gain of 1.

The GI has a remarkable filtering action on wide-band input noise because, since now we can choose any Tg, we are reducing the BW in the frequency domain, and if so we are acquiring less noise.

Time domain and frequency domains

In the time domain we need the autocorrelation of the weighting function, because the value in zero for the autocorrelation, for the white noise, gives us the noise at the output. The amplitude in zero is not 1/Tf, because the area of the filter is not 1, but it is Tg/Tf^2.



Filtering and SNR enhancement by GI

I want to understand the gain from the SNR point of view.

INPUT:
- signal x_s constant in T_G (DC signal)
- wide-band noise S_b (bandwidth $f_n >> 1/T_G$ and autocorrelation width $T_n << T_G$)

$$\overline{x_n^2} = S_b 2 f_n = S_b / 2T_n$$

OUTPUT:

Signal $y_s = x_s \cdot \frac{T_G}{T_f} = x_s G$ i.e. with gain $\boxed{G = \frac{T_G}{T_f} << 1}$

Noise $\overline{y_n^2} = S_b \cdot \frac{T_G}{T_f^2} = \frac{S_b}{T_G} \cdot \left(\frac{T_G}{T_f}\right)^2 = \frac{S_b}{T_G} \cdot G^2 =$

$$= \frac{S_b}{2T_n} \frac{2T_n}{T_G} G^2 = \overline{x_n^2} \cdot \frac{2T_n}{T_G} \cdot G^2$$

Signal-to-noise ratio

$$\left(\frac{S}{N}\right)_y = \frac{y_s}{\sqrt{\overline{y_n^2}}} = \frac{x_s}{\sqrt{\overline{x_n^2}}} \cdot \sqrt{\frac{T_G}{2T_n}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}}$$

VERY IMPORTANT

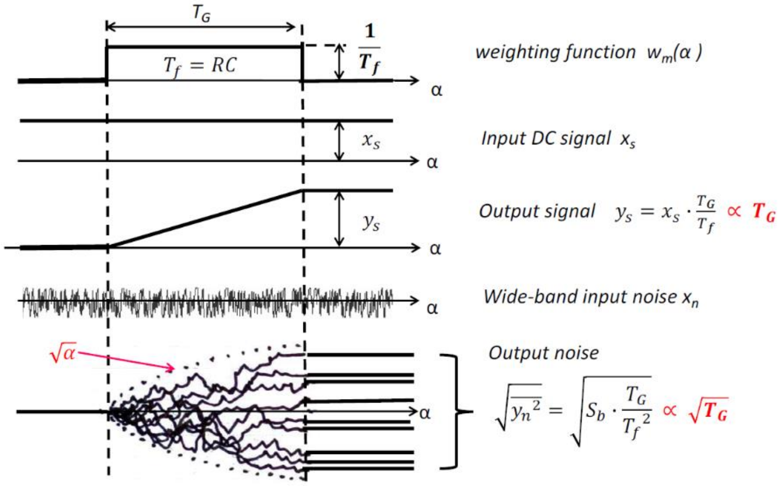
NB: the output signal increases as T_G and the noise as $\sqrt{T_G}$, therefore the S/N increases as the square root of the gate time $\sqrt{T_G}$

The input noise is not the ideal delta, we need the approximation Sb*2*fn (area of the rect in the frequency domain), in the time domain we approximate the delta with a triangle.

For the signal output we have the input signal times the gain (we are integrating the signal, and the gain is the area of the weighting function).
As for the noise, Sb*(Tg/Tf^2), because the noise is Sb times the autocorrelation in 0. Then I do some mathematical tricks to extract the gain, which is a parameter that in the SNR should simplify. Then, Sb/(2Tn) is the input noise, so I can write in the end the output noise as a function of the input noise and of the gain.

The result is that the SNR at the output is the SNR at the input times something, which is the improvement of the signal to noise ratio. It is a key parameter, because we are interested in improving the SNR. We could naively say to increase Tn to increase the SNR, because the noise is a data, we cannot change it. The parameter we can manage is Tg → if we want to increase SNR we have to increase Tg and the SNR improves with the square root of the Tg.

Output signal and noise of GI



An integral of a constant value is a linear value. As soon as I increase Tg I increase the output proportionally to Tg, because I'm picking more signal. As for the noise, **the noise increases with Tg**, but proportionally to sqrt(Tg), so with a ratio of increase that is smaller.
To get an active filter with a gain = 1, Tf = Tg, so the noise goes as 1/Tg.

GI COMPARED TO OTHER LP FILTERS

Fair comparison between different LPF with different DC gain G can be made by considering the value of the **filtered noise referred to the input** of the filter (and the input signal). This is equivalent to consider the **output with unity DC gain** (if necessary, by considering to add further gain stages).
For a GI this noise is

$$\left(\overline{x_n^2}\right)_{GI} = \frac{\left(\overline{y_n^2}\right)_{GI}}{G^2} = \frac{S_b}{T_G}$$

For a constant-parameter RC (inherently with G=1) that filters the same wide-band noise S_b it is

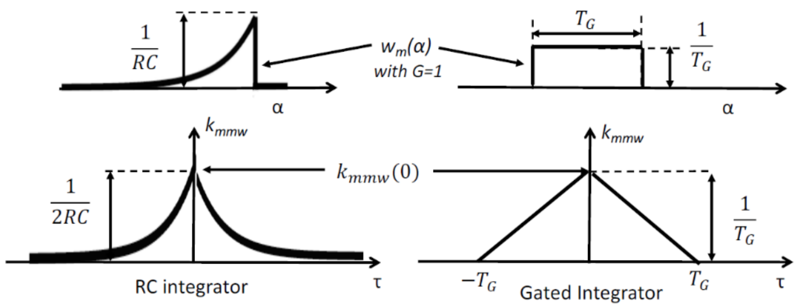
$$\left(\overline{x_n^2}\right)_{RC} = \left(\overline{y_n^2}\right)_{RC} = \frac{S_b}{2RC}$$

Therefore, as concerns the S/N obtained for input DC signals accompanied by wide-band noise, GI and RC integrator are equivalent if

$$T_G = 2RC$$

In making the comparison, we notice that the gain is not 1, but we can normalize the filter using a unitary DC gain for Tf = Tg (active integrator). At this point, the output of the noise is Sb/Tg.
We get, as a final result, that if Tg = 2RC the two filters have the same noise in output.

NB: we cann apply a GI only if we have a sync signal.



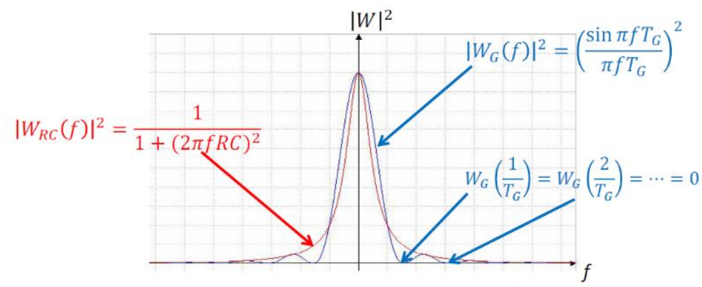
We consider here filters with **equal DC gain of unity**, hence with equal output signal.
With wide-band input noise S_b the output noise is

$$\overline{y^2} = S_b \cdot k_{mmw}(0)$$

therefore, GI and RC have **equal output noise** if

$$\boxed{T_G = 2RC}$$

The following is the plot of the RC (red) and of the absolute value squared of the weighting function of the GI filter (blue). The two are very similar. If we compute the areas of the two we get the same value, because we get the same noise iif we have the same area, the same gain.



With $T_G = 2RC$ they are equivalent for:

- the S/N obtained with wide-band noise and DC signal input
- the attenuation of high-frequency disturbances in general

However:

- **The GI has zeros of $W_G(f)$ at $f_k = k/T_G$ that can be exploited to cancel specific disturbances at known frequencies (radio frequencies or mains frequency and harmonics)**

AVERAGING FILTERS

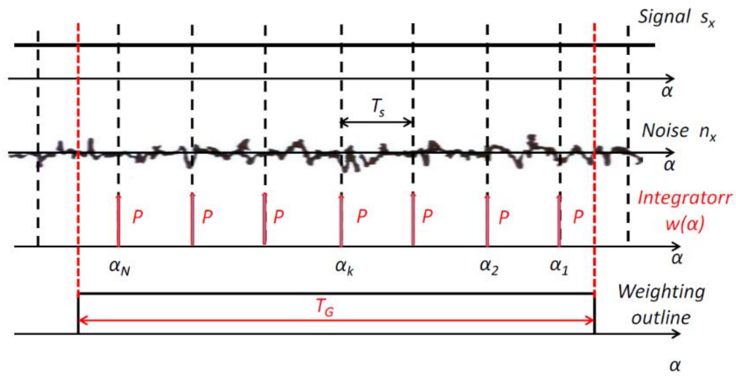
- Discrete time integrator (DTI)
- Boxcar integrator (BI)
- Ratemeter integrator (RI)

DISCRETE TIME INTEGRATOR

We want to study digital filters to compare them with the analog ones. The DTI is the average; from an analog point of view, the average is the GI with amplitude 1/Tg and duration Tg. For the digital average Ts is the distance between two samples, P the weight of each sample and Tg is the width of the equivalent analog filter, which is Tg = N*Ts.

The other hypothesis is that 2*Tn << Ts. The width of the autocorrelation of the noise, 2*Tn, much be much smaller than Ts. If this condition is satisfied, since the autocorrelation ends in a very small time, the samples of the noise are totally uncorrelated in the time domain.

It is the **discrete-time** equivalent of a continuous gated integrator with gate $T_G = N\,T_s$



- Samples taken with sampling frequency $f_s=1/T_s$ i.e. at intervals T_s within T_G
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n << T_s$)
- Every sample is multiplied by P and summed, up to a total **$N = T_G / T_s$ samples**

For the GI, the SNR improved with a factor of sqrt(Tg). Now we want to demonstrate that the SNR of the DI increases with something like sqrt(N).
The output of the signal is the input one multiplied N times the weight P. The DC gain is NP, which is not 1. If we set P = 1/N, it is 1.
As for the noise, we are making the sum also of the samples of the noise (n_xk is the sample we are considering). Then we compute the square value of n averaged on ensembles.

With **white noise**, the GI gives $S/N \propto \sqrt{T_G}$; we show now that the DI gives $S/N \propto \sqrt{N}$

The **output signal** is

$$\boldsymbol{s_y = N \cdot P s_x} \quad \text{(that is, the DC gain is } G = N \cdot P\text{)}$$

The **output noise** is $n_y = \sum_{k=1}^{N} P \cdot n_{xk}$ and

$$\overline{n_y^2} = P^2\overline{({n_{x1}}^2 + {n_{x2}}^2 \ldots + n_{x1}n_{x2} + \cdots)} = P^2(\overline{{n_{x1}}^2} + \overline{{n_{x2}}^2} + \ldots + \overline{n_{x1}n_{x2}} + \cdots)$$

The noise samples are not correlated

$$\overline{n_{x1}n_{x2}} = \overline{n_{x2}n_{x3}} = \cdots = 0$$

and the noise is stationary $\overline{{n_{x1}}^2} = \overline{{n_{x2}}^2} = \cdots = \overline{{n_x}^2}$

Therefore

$$\boldsymbol{\overline{{n_y}^2} = N \cdot P^2\overline{{n_x}^2}}$$

By summing N samples the signal is increased by N and the rms noise by $\sqrt{N}$
The SNR is thus improved by the factor $\sqrt{N}$

$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{{n_y}^2}}} = \frac{N \cdot P s_x}{\sqrt{N \cdot P^2\overline{{n_x}^2}}} = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

If noise samples are not correlated, cross products are zero. Furthermore, if the noise is stationary, the square vale is the same.

Finally we ca get the SNR_out, which depends on the sqrt(N).

Discrete time averager

We want to compare the digital and analog approach, so we need to set the same gain and equal to 1. Therefore, the signal is the same.

An averager is simply a discrete-time integrator with sampling weight *P* adjusted to give unity DC gain, that is $G = N \cdot P = 1$

$$P = \frac{1}{N}$$

and therefore output signal equal to input

$$s_y = s_x$$

The output noise is reduced to

$$\overline{n_y{}^2} = N \cdot P^2 \overline{n_x{}^2} = \frac{1}{N} \cdot \overline{n_x{}^2}$$

$$\sqrt{\overline{n_y{}^2}} = \frac{\sqrt{\overline{n_x{}^2}}}{\sqrt{N}}$$

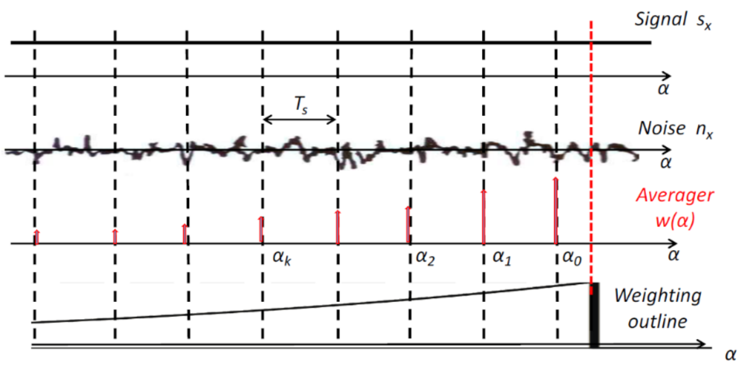
which corresponds to the enhancement of the S/N

$$\left(\frac{S}{N}\right)_y = \sqrt{N} \cdot \left(\frac{S}{N}\right)_x$$

N = Tg*fs, and the improvement of the SNR is sqrt(N), so sqrt(Tg/Ts). In the gate integrator it was sqrt(Tg). It seems that if we change Ts we can obtain any improvement, we have just to increase the freqeuncy. So it seems that the digital sampling could be much better than the analog one, just we need to increase fs. But the problem is that if we increase fs, it seems that we are increasing sqrt(N) (if stationary noise and noise samples uncorrelated, if neither of the two hypothesis is valid the formula is not true), but samples start to be correlated.

Example – exponential averager

It is the discrete-time equivalent of an RC integrator



- Samples are taken with sampling frequency $f_s=1/T_s$ i.e. at intervals T_s
- Input: DC-signal s_x and wide-band noise n_x (autocorrelation width $2T_n << T_s$)
- The sample **weight slowly decays** with the sample «age»: $w_k = Pr^k$ **with $(1 – r) << 1$** X

Why don't we try to compare also the RC with a digital filter? Can we create something that recalls the RC weighting function in a digital approach?

NB: in this example the signal is constant, at the exam it might not be.

A digital approach is to use a sampling with a weight that goes down with a power law. Why should we choose the equal weights situations or the decreasing weight situation?

If the signal is constant, both the solutions can be used and we will choose the one that maximizes the SNR. If the signal is not constant, but it is neither a very fast signal (e.g. the temperature of a room), so it changes slowly over time, to maintain the noise uncorrelated we can use only a certain amount of samples. The idea is to take more samples over a larger time giving a high weight at the samples near to where I'm observing the situation, and less weight to the samples back in time, which are less correlated with the current temperature. This is better than using a lot of samples in a short window close to t_m.

If instead the signal changes as a square wave, it is instead better to use constant weight if I need to sample each change time interval. The idea of the exponential decay time is to increase the amount of time for the average but if the signal changes over this time we decrease the weight of the last part of the filter. If the signal is constant it is better to weight all the samples with the same weight.

Now we can make the computations (previous image). The low is the x one, and it is a power law with a specific constraint.

For the signal, we have to sum all the samples, but each sample with the correct weight. The sum from 0 to infinite of r^k is 1/(1-r), so the DC gain is P * 1/(1-r).
Same approach for the noise. Also here we can use the two hypothesis on the stationary noise and uncorrelation of samples.

Output signal $s_y = s_x \cdot P \cdot \sum_{k=0}^{\infty} r^k = s_x \cdot P \frac{1}{1-r}$ y (i.e, DC gain $G = P\frac{1}{1-r}$)

Output mean square noise

$$\overline{n_y^2} = P^2 (\overline{n_{x0}^2} + r^2 \cdot \overline{n_{x1}^2} + \dots + r^{2k} \cdot \overline{n_{xk}^2} + \dots + r^k r^j \cdot \overline{n_{xk} n_{xj}} + \cdots)$$

The noise samples are not correlated ($\overline{n_{xk} n_{xj}} = 0$ for $k \neq j$)

and the noise is stationary ($\overline{n_{x0}^2} = \overline{n_{x1}^2} = \cdots = \overline{n_x^2}$)

Therefore

$$\overline{n_y^2} = \overline{n_x^2} \cdot P^2 \left(1 + r^2 + \dots + r^{2k} + \dots\right) = \overline{n_x^2} \cdot P^2 \cdot \frac{1}{1-r^2}$$

The SNR is thus improved to

$$\left(\frac{S}{N}\right)_y = \frac{s_y}{\sqrt{\overline{n_y^2}}} = \frac{P s_x}{1-r} \frac{1}{\sqrt{\frac{\overline{n_x^2} P^2}{1-r^2}}} = \left(\frac{S}{N}\right)_x \sqrt{\frac{1+r}{1-r}} \quad \text{x}$$

But the attenuation ratio r is very close to unity ($1 - r) << 1$ hence ($1 + r) \approx 2$ and therefore

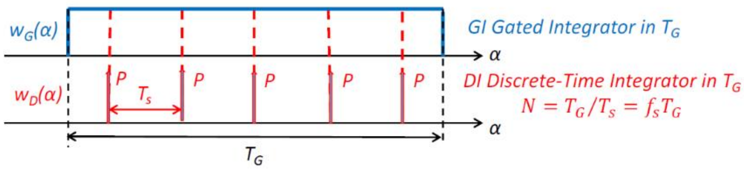
$$\left(\frac{S}{N}\right)_y \cong \left(\frac{S}{N}\right)_x \sqrt{\frac{2}{1-r}}$$

The condition 1 – r << 1 is used in x. The result is the blue formula. This expression is no more valid if we don't have an input signal that is constant, because the noise part is the same for any signal, but the signal part is different. If the signal is changing we have to take into account its change in the expression y.

DISCRETE TIME INTEGRATOR VERSUS GI

The idea is to show which is the best, that will be the analog filter. It is more intuitive to understand this in the frequency domain rather than in the time domain.

Firstly we define our system. For us, a digital filter is an analog filter. The first contender is the GI, and the digital filter is a series of delta, but a delta is an analog filter. It is important to consider a digital filter as an analog one because we don't know anything so far about digital filters, and we have all the formulas for the analog filters.
If I want to compare in a fair way the two filters, I need to set the same gain, and the best solution is 1. So the amplitude of the GI is 1/Tg, and P = 1/N.



INPUT: DC signal s_x and wide-band noise S_b (bandwidth $2f_n >> f_s$, correlation width $2T_n << T_s$) with rms value $\overline{n_x^2} = S_b 2f_n = S_b/2T_n$

With unity DC gain $s_y = s_x$

- Noise reduction by GI $\sqrt{\overline{n_{yG}^2}} = \sqrt{\overline{n_x^2}} \Big/ \sqrt{\frac{T_G}{2T_n}}$
- Noise reduction by DI $\sqrt{\overline{n_{yD}^2}} = \sqrt{\overline{n_x^2}} \,/\, \sqrt{N}$

The other hypothesis is that **we consider the BW of the noise much larger than fs** → we consider the noise white, so that we can use Sb/2Tn.
If we set a unity DC gain, for the Gi the improvement of SNR is the blue expression, and for the DI it is the red one.

The improvement factor is

- $\sqrt{N}$ for the DI, increasing with the number N of samples taken
- $\sqrt{T_G/2T_n}$ for the GI, constant for a given T_G

QUESTION : is it possible to attain with a DI better S/N improvement than a GI just by increasing the number N (i.e. by using very fast sampling electronics)?

ANSWER: NO !!

In fact , since $N = T_G/T_s$ for having $N > T_G/2T_n$ it must be

$$T_s < 2T_n$$

in these conditions

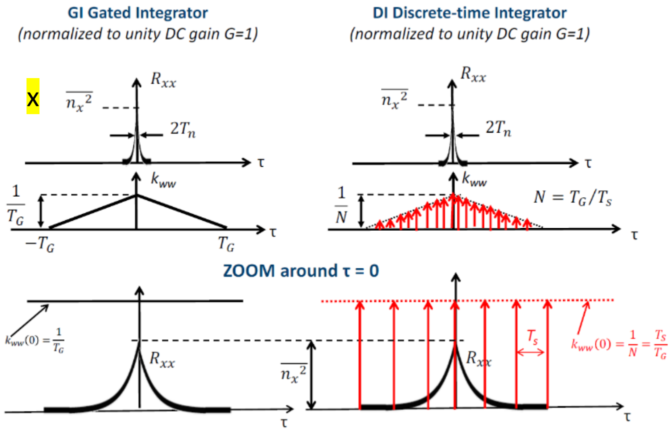
- the samples are no more uncorrelated
- the improvement factor is **no more given by $\sqrt{N}$**
- There is still an improvement factor, but it must be evaluated taking into account the correlation between the noise samples.
- It is anyway $(S/N)_{DI} \leq (S/N)_{GI}$ with $(S/N)_{DI} \rightarrow (S/N)_{GI}$ as N is increased, as we can demonstrate in time domain and in frequency domain

Since one of the digital filter goes with sqrt(N) and the other one as sqrt(Tg/2Tn), it seems that we can obtain any improvement of the digital filter with N. In theory also if Tg goes to inf the SNR improves, but to do so we need the signal to span to infinite values, but the signal is typically limited in time. Moreover, a large Tg means that we need a lot of time to wait for the filter. So it seems easier to increase N for the DI, but also here just increasing N is not improving SNR because at a certain point it is no more correct that the samples are uncorrelated.

If Ts < 2Tn, so if we sample noise that starts to be correlated, we still have an increase of SNR, but the improvement factor is no more sqrt(N).
The limit of the improvement of the SNR will be the SNR of the analog filter. In fact, if we increase the number of delta we are approximating the analog filter that is the rect.

Time domain

We have to compare two different filters. The first thing to do is plotting the weighting function of both filters and the autocorrelation (in time domain the product of the autocorrelation of the filter and of the noise integrated from -inf to +inf gives us the output noise in the time domain). For the GI (or MM filter) we have the autocorrelation k_ww of the filter and Rxx the one of the noise.



The area of the delta x is the spectral density, Sv, times the value in 0 of the autocorrelation of the weighting function in the frequency domain.
The weighting function of the DI is a series of delta. The autocorrelation of the comb of delta is another series of delta, but how many and with which shape? The number of deltas is doubled and with a triangular shape. The value in 0 is 1/N, which is (1/N)^2 * N (we are summing N delta in 0, so it is not (1/N)^2). At this moment we are not interested in the shape of the autocorrelation, we just want the value in 0.

Let's zoom around 0, the autocorrelation of the filter in the analog domain is flat, because the triangle is very flat, and the value is k_ww(0). As for the digital filter, we will find a set of deltas of amplitude 1/N, still k_ww(0). The number of deltas we find around 0 should be 1, because the samples are uncorrelated and since the delta tells us where we are sampling, one delta is in 0 and the other where the autocorrelation is 0 (check the book).
In our case delta are no more decreasing like a triangle because we are decreasing Ts at so high level that the number of delta is so high that there is indeed a decrease as a triangle, but the triangle is so huge that at first approximation the deltas around 0 have the sample amplitude (similar to the analog case).

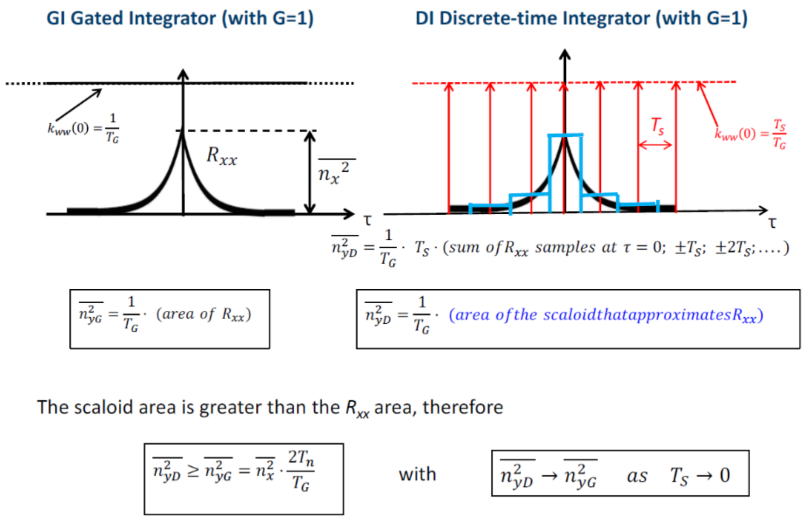
Now we need to perform the computations. How can we compute the noise (gain is the same so I need to compare just the noise)?
We need to multiply the autocorrelation of the noise, the autocorrelation of the filter and make the integral, and we want to do this both in the analog and digital domains. In the digital domain the integral will be only in some points.

For the digital we have to multiply the series of delta times the autocorrelation, and if we multiply a series of delta with an analog shape, we have some deltas with the value of the autocorrelation, and then we have to make the sum. The amplitude of the delta is Ts/Tg. Then, Ts times the samples of the filter's autocorrelation are the area of the blue box of the following image.

This is the same result for the analog, where the area of Rxx is the result of the integral.

Now, if we reduce Ts, we are reducing the base of the boxes, so approximating the analog approach, so getting the same noise of the analog filter and since the signal is the same, also the same SNR (if Ts → 0).

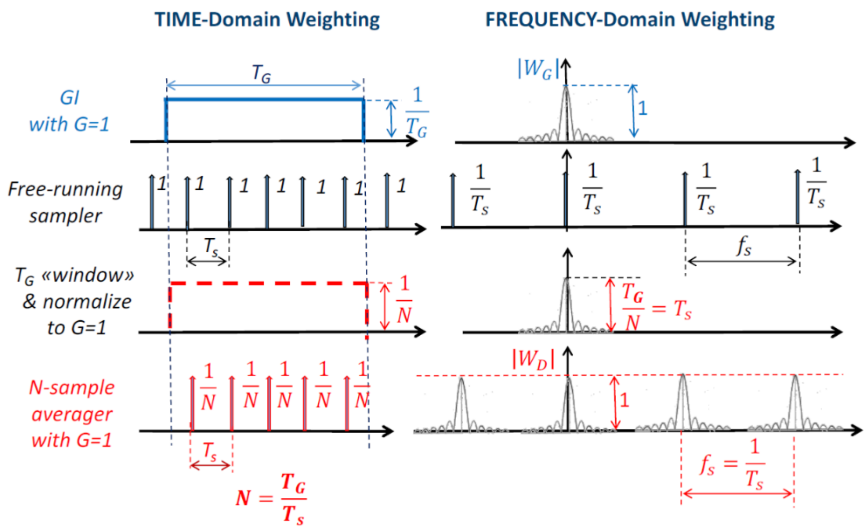
The problem is that, ok the limit of the digital filter is the analog one, but it seems that for any value of Ts the digital filter has the SNR smaller than the analog one, so the area of the sum of the blue boxes is the area of the autocorrelation. This is a correct sentence, but it is difficult to be demonstrated for each Ts (check book).



Hence we move in the frequency domain where we don't need any demonstration for this.

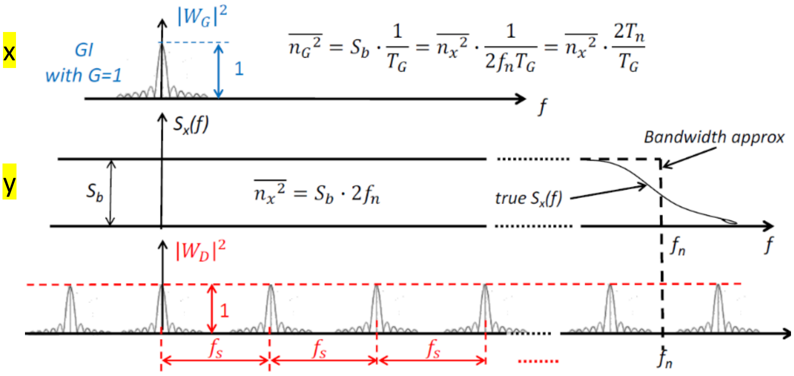
Frequency domain

The output noise in the frequency domain is the integral of the PSD of the noise multiplied by the absolute value squared of the Fourier transform of the weighting function, from -inf to +inf. Let's do this for the analog and digital filters.



In the frequency domain, the Fourier of the rect is the sinc with a zero in 1/Tg and value in 0 that is 1, which is the area of the rect (Tg * 1/Tg).
As for the digital filter, the Fourier transform of a comb of delta is a comb of delta. But we don't have an infinite number of deltas, but a finite one. To get a finite number of delta, we multiply the infinite comb of delta with the rect in the time domain, so in the frequency domain I need to convolve.
Hence in the frequency domain we will have the convolution of the Fourier transform of the rect in the position of the deltas. We can check if the value in 0 is still 1 because the value in 0 of the digital filter was with area 1 in the time domain.

Now we have the Fourier transform of the digital filter. We multiply by the PSD of the noise (Sb) and make the integral.



$$\overline{n_G^2} = \int_{-\infty}^{\infty} S_x(f)\, |W_G(f)|^2 df \qquad \overline{n_D^2} = \int_{-\infty}^{\infty} S_x(f)\, |W_D(f)|^2 df$$

The figure illustrates how the output noise $\overline{n_D}^2$ is reduced and S/N is enhanced by increasing the sampling frequency f_s (for a given averaging time T_G)

The integral from -inf to +inf of the product of x and y is the area of x. The digital filter has Sb limited by fn, In the digital filter we have an infinite number of replicas of the weighting function, but we have to consider the replicas up to fn.

At first approximation, the noise of the analog is the area of one sinc, while the noise of the digital is the area of five sinc (from the image). So the noise of the digital filter has to be larger than the analog one, because we are integrating more replicas.

How can we demonstrate now that the limit of the digital filter is the analog one?
As soon as I increase fs (so I reduce Ts) I'm pushing some replicas outside the limit of the noise, so ideally we reach the output noise of the analog filter.
So when fs is larger than fn we have just one replica, but we are not so lucky because the sinc has a tail that spans to infinite, so to get exactly the same value we need to push the replica to infinite distance. So also in the frequency domain I can say that the limit of the digital approach is the analog filter.

Noise filtering analysis: GI vs DI

a) As long as $f_s \ll f_n$:

- the noise samples are uncorrelated
- each line of $|W_D|^2$ is identical to $|W_G|^2$ of the GI (with same DC gain G=1)
- a high number N_L of lines of $|W_D|^2$ falls within the noise bandwidth $2f_n$
- the output noise of the DI is N_L times that of the GI

$$\overline{n_D^2} = \overline{n_G^2} \cdot N_L$$

With good approximation it is

$$N_L \cong 2f_n / f_s$$

and it is confirmed that for uncorrelated samples the S/N increases as $\sqrt{N}$

$$\overline{n_D^2} = \overline{n_x^2} \cdot \frac{1}{T_G f_s} = \frac{\overline{n_x^2}}{N}$$

b) When f_s becomes comparable to f_n or higher

- the previous result is no more valid.
- the output noise must be computed with the actual noise spectrum

$$\overline{n_D^2} = \int_{-\infty}^{\infty} S_x(f)\, |W_D(f)|^2 df \geq \overline{n_G^2}$$

- The figure shows that $\overline{n_D^2}$ is always higher than $\overline{n_G^2}$ and attains it for $f_s \to \infty$

$$\lim_{f_s \to \infty} \overline{n_D^2} = \overline{n_G^2}$$

We have a DC gain = 1 so we have to make the integral of the number of replicas that fall in fn, the bandwidth of the noise. This number can be computed saying that the output noise is the analog one (i.e. the integral of one replica) times the number of replicas. The number of replicas N_L = 2* fn/fs.

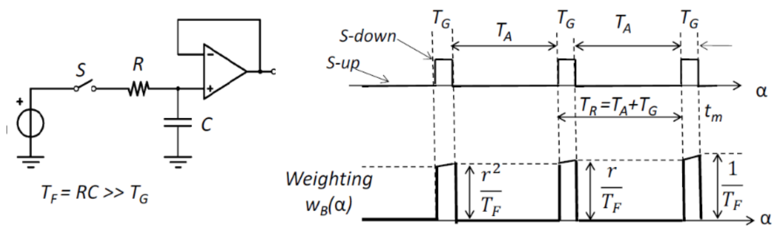
In the end we must have the same result in the frequency domain that we got in the time domain.

BOXCAR INTEGRATOR – BI

This simple analog circuit combines two functions:

1. Sample Acquisition by gated integration
2. Exponential **averaging of samples**

The circuit employed is the same of the Gated Integrator, but with a fundamental difference: the capacitor is **NOT RESET between the acquisitions.**



- In T_A the C is in HOLD state: nothing changes, no memory loss and no new charge input
- In T_G the discharge of C (memory loss) reduces the previously stored value by the factor $r = e^{-T_G/T_F}$. **NB:** r does NOT depend on the interval T_A

We want to combine the digital and analog approaches. We will acquire the signal with the GI but, instead of acquiring it one time, we acquire it several times and make a digital average of the acquired signals. This because sometimes just one acquisition of the signal is not enough, because its amplitude is too small. This is not possible always, but e.g. for the fluorescent emitted by a single molecule this can be done. For the satellite signal it is the same, since it is at a fixed distance, we can send a pulse and repeat the measurement several time.

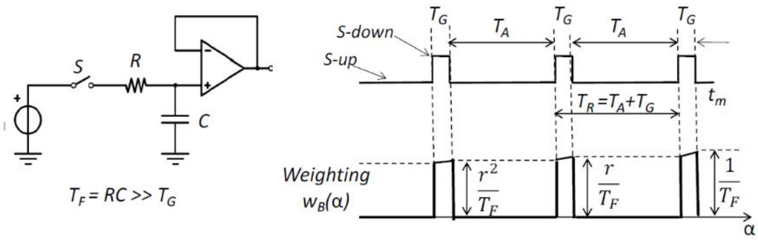
The idea is to use just the GI (circuit on the left, RC and a switch, the buffer is not compulsory) and removing the reset of the capacitor from it to get the GI. In fact, one of the hypothesis of the GI was to start with a capacitor empty for every measurement. In our new system we remove this hypothesis.
Since it is a NCPF, we need to define how to control the switch. We do it periodically, without discharging the C.

With a new filter, the first thing to dimension is the weighting function. We create it with the usual procedure. Instead of closing just one time the switch I open it and close it without the reset. I apply a delta every time the switch is closed to create the weighting function. The capacitor can discharge only when the switch is closed, so from +inf to -inf the capacitor is discharging (theoretically exponentially).

The other important thing when defining a new filter is the autocorrelation of the noise and gain.
From the gain standpoint, we are integrating the exponential decay time, but only some slices (it is sliced because of the action of the switch) of it so the area will be lower than the full area of the exponential decay time.
The gain is the value in 0 in the frequency domain, that is the integral from -inf to +inf of the weighting function in the time domain. But the w_f is the exponential decay time just sliced and split, so the area is exactly the original area of the exponential decay time, which is 1 in a standard RC. So the gain is 1.

The other parameter is the autocorrelation, theoretically. The autocorrelation is a function, but at first level, if we consider white noise, we are interested in the value in 0. Hence the question is: which is the value in 0 of the w_f?
We need to take the function, multiply it by itself and take the integral. The value in zero is exactly the value in 0 of the autocorrelation of the non-sliced exponential. So the autocorrelation in zero is 1/2Tf.

We won't compute the autocorrelation function of the boxcar because it is not useful.
Now we everything to compute the improvement in SNR.



- BI behaves as RC-integrator (RCI) when the switch is closed (S-down); it is in HOLD state when the switch is open (S-up)
- In fact, the weighting function $w_B(\alpha)$ of the BI is obtained by subdividing $w_{RC}(\alpha)$ of the RCI it in «slices» of widthT_G and placing them over the S-down intervals
- G=1 : the DC gain of BI (area of w_B) is unity (like that of RCI): the BI is an averager
- The autocorrelation functions k_{wwB} of BI and k_{wwRC} of RCI are very different, but have equal central value $k_{ww}(0)$

$$k_{wwB}(0) = k_{wwRC}(0) = \frac{1}{2RC} = \frac{1}{2T_F}$$

SNR ENHANCEMENT IN THE BI

The approximation of the white noise in the time domain is Sb/2Tn, this is a data. Then we are interested in the output of the BI, knowing k_wwB(0).

The input wide-band noise S_b with bandwidth $2f_n$, autocorrelation width $2T_n$, has mean square value

$$\overline{n_x{}^2} = S_b \cdot \frac{1}{2T_n}$$

The BI output noise is

$$\overline{n_y{}^2} = S_b \cdot k_{wwB}(0) = S_b \cdot 1/2T_F = \overline{n_x{}^2} \cdot \frac{T_n}{T_F}$$

Therefore, since BI has G=1 the S/N enhancement is

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

The S/N enhancement does **NOT** depend **on the RATE** of the samples because it is obtained by **averaging over a given number** of samples and **not over a given time interval**. In fact, counting the samples (from the measurement time t_m and going backwards) the sample weight is reduced below 1/100 for sample number > $4.6T_F/T_G$, irrespective of the sample rate

The SNR in output is the input SNR times sqrt(Tf/Tn). The signal in input is exactly equal to the signal in output, we can reason only on the noise.

The SNR enhancement doesn't depend on the rate, and in fact the frequency of the pulses is not in the formula.

In the end we are making an average with exponential weight of a lot of GI.
So the improvement in SNR of the GI is sqrt(Tg/2Tn); but we are making the average of different GI exponentially, and the exponential averaging has an improvement factor of sqrt(2Tf/Tg). Putting them together we get the improvement factor of the BI.

However, there are some applications where we want to have a dependance on

The BI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the BI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_G/T_F} \cong 1 - T_G/T_F$$

which enhances the S/N by the factor

$$\sqrt{(1+r)/(1-r)} \cong \sqrt{2/(1-r)} = \sqrt{2T_F/T_G}$$

NB: this factor is INDEPENDENT of the RATE of samples, because the AVERAGE IS DONE ON A GIVEN NUMBER OF SAMPLES and not on a given time.

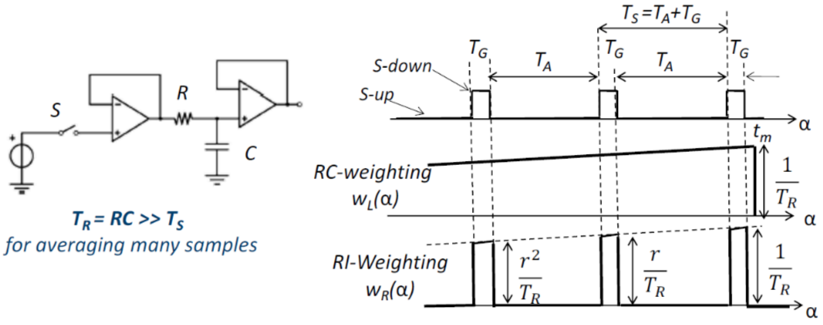
The S/N enhancement is thus confirmed and clarified

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_F}{T_G}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_F}{T_n}}$$

the ratio, because sometimes the information on the signal is not in the amplitude or shape, but in the ratio.

RATEMETER INTEGRATOR – RI

The circuit is the same, but I add a buffer between the switch and the RC. If we compute the w_f, when the switch is closed we get the RC, but when it is open, the RC sees the low impedance of the output of the buffer, so the C can discharge through this path. So the capacitor keeps discharging, doesn't matter if we open or not the switch.

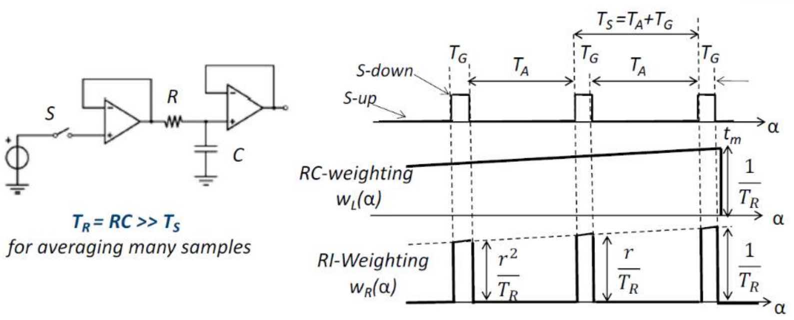


- By inserting a buffer between S and RC a new **exponential averager** is obtained, radically **different from BI**. The integrator is no more a switched-parameter RC filter: it is now a constant-parameter RC filter, unaffected by the switch S.
- There is no HOLD state. The memory loss goes on all the time; the weight reduction from sample to sample is $r = e^{-(T_G+T_A)/T_F} = e^{-T_S/T_F}$. **NB:** r **DEPENDS** on the sample **RATE**!
- During T_G (with S-down) the input is integrated in C
  During T_A (with S-up) the input is NOT allowed

Again, we want to retrieve the w_f, the gain and the autocorrelation.

The w_f is the same as in the BI, but the second slice doesn't start when the previous has finished, because the capacitor has discharged in the meantime.
As for the gain, it is less than 1 because we are making the integral only of some slices of the fully exponential decay time, because when the w_f is 0, the capacitor is still discharging.
The problem is that computing the autocorrelation in 0 of the RI w_f is not easy, because it is no more the exponential sliced.

So what we can do is to see the RI as the product of a lot of GI (if Tg << Tf) and the average of exponential decay time, but the difference is that the amplitude difference between one GI and the other is no more Tg/Tf, but (Ta + Tg)/Tf.



- The **DC gain is G < 1** (the RC filter has G=1, but it receives just a fraction of the input!)
- With $T_R >> T_S$ the DC gain G is proportional to the sample rate $f_S = 1/T_S$

$$G = \int_{-\infty}^{\infty} w_R(\alpha) d\alpha \cong \frac{T_G}{T_S} \cdot \int_{-\infty}^{\infty} w_L(\alpha) d\alpha \cong \frac{T_G}{T_S} = f_S \cdot T_G$$

NB: if the input signal amplitude x_S is constant but f_S varies, the output signal y_S varies.
In fact, the circuit is also employed as **analog ratemeter**: with constant input voltage x_S it produces a quasi DC output signal proportional to the repetition rate f_S

We want to compute the area of the slices. We know the distance Ts between two slices Ts, the amplitude of the slice that is Tg, and the area of the exponential decay time that is 1. Tf is the exponential time constant. If **Tf >> Ts**, then Tf >> Tg, and every Ts I'm interested in acquiring one slice. Since Tf >>Ts, we have like a rect in of width Ts an amplitude of the exponential decay time. The fraction of the area that we are acquiring every slice is Tg/Ts, under the **approximation of flat curves** and I repeat this for all the Ts. Then the sum of all the recto of base Ts is the exponential decay time, which is one. But each Ts I don't want all the rectangles of width Ts, but the one of with Tg. So in the end the area is Tg/Ts.

SNR ENHANCEMENT

The RI is equivalent to the cascade of two filtering stages

a) Acquisition of samples by a GI with same T_G and T_F as the RI, which enhances the S/N by the factor

$$\sqrt{T_G/2T_n}$$

b) Exponential averaging of the samples with attenuation ratio

$$r = e^{-T_S/T_R} \cong 1 - T_S/T_R$$

which enhances the S/N by the factor

$$\sqrt{\frac{1+r}{1-r}} \cong \sqrt{\frac{2}{1-r}} = \sqrt{\frac{2T_R}{T_S}} = \sqrt{2T_R f_S}$$

NB: this factor DEPENDS on the sample RATE f_S because the AVERAGE IS DONE ON A GIVEN TIME and not on a given number of samples. The weight reduction is below 1/100 for samples that at the measurement time t_m are «older» than 4.6·T_R

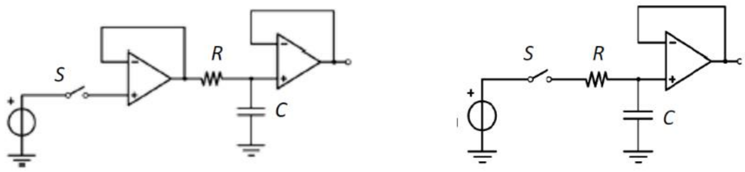
The S/N enhancement thus depends on the sample rate f_s

$$\left(\frac{S}{N}\right)_y = \left(\frac{S}{N}\right)_x \cdot \sqrt{\frac{T_G}{2T_n}} \cdot \sqrt{\frac{2T_R}{T_S}} = \left(\frac{S}{N}\right)_x \cdot \sqrt{f_S T_G \frac{T_R}{T_n}}$$

Point b) is the improvement of the digital average.
The final SNR improvement formula depends on the frequency fs. If we are interested in measuring the rate, this is a good thing. Instead, if we are not interested in the rate, because it is e.g. fixed, at this point using a RI is useless. In fact, if for some reasons the rate is changing statistically and we are not interested in the rate, we cannot use this formula because we don't get something reasonable.

To get if the BI or the RI is better, in both cases the GI is the same, so its ratio of improvement is fixed, while we have to compare the digital average of the exponential decay time for the BI and for the RI. But in both cases we are choosing Tf to maximize the SNR improvement.

Passive circuit comparison – BI and RI



RATEMETER INTEGRATOR

- Switch S acts as gate on the input source
- Switch S is decoupled from the RC passive filter by the voltage buffer
- The RC integrator is unaffected by S, it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the RC value

BOXCAR INTEGRATOR

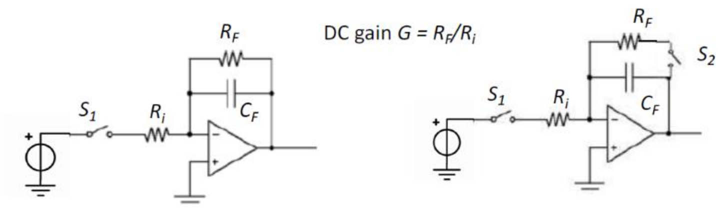
- Switch S acts as gate on the input source
- Switch S acts also on the RC passive filter (changes the resistor value)
- The time constant T_F of the integrator filter is switched from finite RC (S-down) to infinite (S-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value

In both cases we have a switch, so in order to control it we need to know where the signal is → we need the sync signal, compulsory.

Tf is a value we need to choose to understand what I want from the signal. The BI doesn't depend on the Tf, but on how many times I open and close the switch. In fact, with the ratemeter, after a certain time there is nothing to integrate, the capacitor has discharged. Instead, with the BI I discharge the capacitor only when I acquire the signal.

Active circuit comparison – BI and RI

Since in the RI we have a gain smaller than 1, the solution is to introduce an active circuit with a gain. these two active filters are identical to the passive ones, because, for us, the gain doesn't matter.



RATEMETER INTEGRATOR

- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- The R_FC_F integrator is unaffected by S_1; it has constant parameters, it does NOT have a HOLD state
- The sample average is done on a given time, defined by the R_FC_F value

BOXCAR INTEGRATOR

- Switch S_1 acts as gate on the input
- Switch S_1 is decoupled from the active RC integrator by the buffer action of the OP-AMP virtual ground
- A second switch S_2 is required for switching the time constant T_F of the integrator from finite R_FC_F (S_2-down) to infinite (S_2-up, HOLD state)
- The sample average is done on a given number of samples, defined by the T_F/T_G value

OPTIMUM FILTER

Which is the best possible filter in terms of SNR? It depends on the information we want to extract.

AMPLITUDE MEASUREMENT OF PULSE SIGNAL

When the signal is really small, normally we are not interested in the shape of the signal (which is known generally), but in the amplitude and in the area. If we know the shape of the signal, speaking about the area or the amplitude is the same, because from one we can compute the other if we know the shape.

Examples

Pulse signals can carry information, which in many cases is contained in the **amplitude** of the pulse, not in the pulse shape or in other parameters (e.g. pulse risetime, duration, etc). These cases can be better illustrated by a couple of examples

- **Automated analysis of biological cells**
  Fluorescence methods are often employed. A cells in diluted solution are labeled with a fluorescent dye that attaches specifically to a given component of the cell. The cells are conveyed by a laminar stream in a small duct and cross a laser beam that excites their fluorescence. The fluorescence pulse emitted by a cell has **intensity** proportional to the **quantity of component** in the cell. By measuring and classifying many pulses (i.e. by collecting the measurement histogram) the distribution of the component in the cell population is obtained.
- **Ionizing radiation spectrometry**
  The radiation detectors generate for each quantum of radiation received (e.g. a Gamma ray) a current pulse with **charge** proportional to the quantum **energy**. By measuring the charge of each pulse and collecting the histogram of measurements, the radiation distribution in energy is obtained (the energy spectrum). It is thus possible to identify radionuclides in the source (e.g. Plutonium in the elements of a nucler reactor); to measure their quantity; to monitor radiation doses etc.

Ionizing radiation spectrometry

It is a typical case of pulse-amplitude measurement , let us consider it in more detail

- Detectors of ionizing radiation (e.g. Gamma rays) generate a current pulse signal for each radiation quantum received
- The **charge** of the pulse signal is proportional to the radiation quantum **energy**
- **The electronics has to measure the charge of each pulse**, not its shape or position in time. In fact, the initial time of each pulse is known (signaled by auxiliary electronics) and all pulses have equal shape, i.e. equal waveform with normalized amplitude
- **The precision of the measure is limited by noise sources in the detector** and in the electronics
- Measurements of many pulses are collected and **classified by size**
- **The distribution of the measured pulse-charge reflects the distribution in energy** (spectrum) of the radiation. The energy scale can be calibrated by measuring radiations with known energy.
- The energy spectrum of the radiation **gives information about the radioactive source** (type and quantity of radionuclides, etc.)

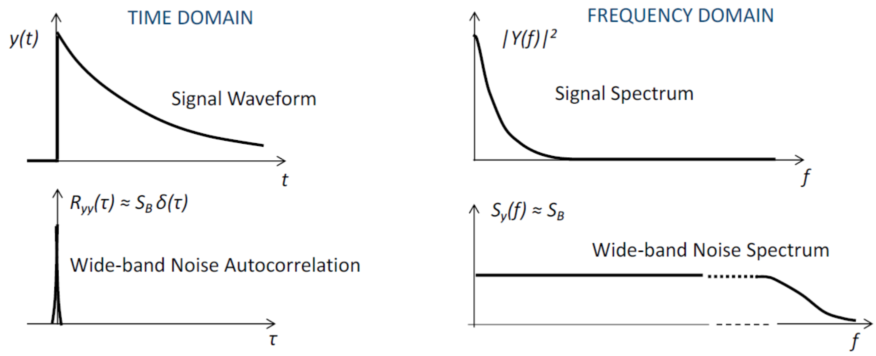
FILTERING TO MEASURE AMPLITUDE OF SIGNALS BESET BY WHITE STATIONARY NOISE

We know the shape but we don't know the area and amplitude, but we have a LF signal, so if we compare the signal with the autocorrelation of the noise, the autocorrelation of the noise is much smaller than the one of the signal, so we can assume the noise as white (i.e. in the frequency domain its BW is more extended than the one of the signal).

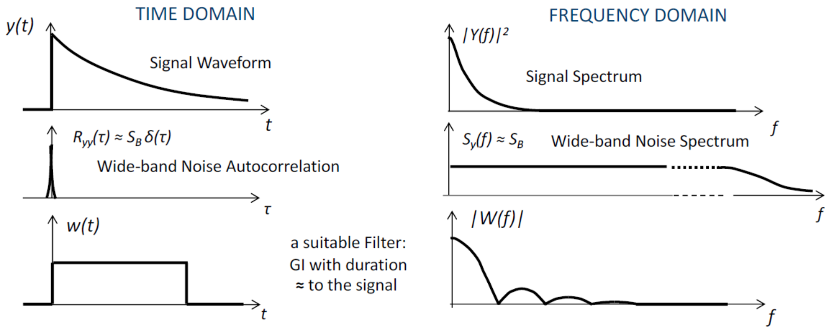
In the frequency domain it means that the BW of the noise is much larger than the BW of the signal.

Let's consider first a basic case: **pulse signals** accompanied by **stationary white noise**.
This ideal case is a good approximation for **real cases** where pulse signals are accompanied by wide-band noise, i.e. noise with

- Narrow autocorrelation, i.e. **width much smaller than the signal duration**
- Wideband uniform spectrum, i.e. **upper bandlimit much higher than that of the signal**



The filter we need to design is probably a LP filter, because the information in the frequency domain is around 0. This is a rect in the time domain, for instance. The question is: is it possible to design the best possible LP filter?



For a measurement of pulse amplitude, a filter has to collect most of the signal and reject most of the noise. It's intuitive that its action should be:

- as seen in time, more or less to average the signal and the white noise over the time interval occupied by the signal
- as seen in frequency, more or less to pass the low frequency range occupied by the signal and cut the higher frequency range where only white noise is present

This means that it's a **low-pass filter (LPF) tailored to the signal** (see GI example above)

The only information that we have is the shape of the signal. We can write the signal as y(t) = A*b(t), where b(t) is a function with an area of 1, so A is the amplitude or area.
One information is the shape of the signal, the other information that we have is that the noise is white.

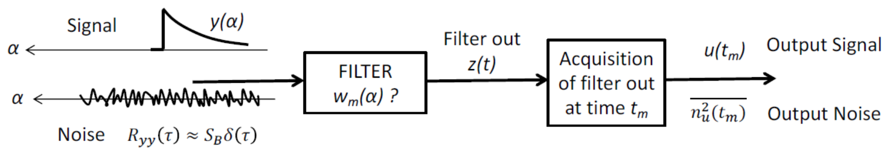
We want to compute the optimum filter theoretically. The signal is defined as y(t) = A*b(t), where A is the area of the signal, and we know the shape of the signal, b(t).
The goal is maximize the SNR.

Low-pass filters tailored to the signal are suitable, but we'd like to know more, since basic questions are still open:

- *is there an optimal filter and if yes, what is it?*
- *If yes, what is the **best obtainable result**? That is, what is the optimized S/N and what is the smallest measurable amplitude?*

The issue is to find out the optimal weighting function, since it completely characterizes a linear filter.
Let's set in evidence the signal area A and the normalized waveform *b(t)*

$y(t) = A \cdot b(t)$ with $\int_{-\infty}^{\infty} b(t)\,dt = 1$



QUESTION: is there a weighting function $w_m(\alpha)$ that optimizes $\frac{S}{N} = \frac{u(t_m)}{\sqrt{\overline{n_u^2(t_m)}}}$ **?**

OPTIMUM FILTERING OF SIGNALS IN WHITE NOISE

The signal at the output is the integral of the product between the weighting function and the signal itself. As for the noise, it's the integral of the autocorrelation of the noise times the autocorrelation of the filter. Since we are considering WN, the autocorrelation of the noise is replaced with a delta.

For the signal, the integral can be rewritten considering the cross-correlation multiplied by the area.

The signal and noise acquired in the measurement are

z $\quad u(t_m) = \int_{-\infty}^{\infty} y(\alpha)\, w_m(\alpha)\, d\alpha = A \cdot \int_{-\infty}^{\infty} b(\alpha)\, w_m(\alpha)\, d\alpha = A \cdot k_{bw}(0)$

z $\quad \overline{n_u^2(t_m)} = \int_{-\infty}^{\infty} R_{yy}(\alpha)\; k_{ww}(\alpha)\, d\alpha = S_B \cdot \int_{-\infty}^{\infty} w_m^2(\alpha)\, d\alpha = S_B \cdot k_{ww}(0)$

Therefore z $\quad \left(\frac{S}{N}\right)^2 = \frac{u^2(t_m)}{\overline{n_u^2(t_m)}} = \frac{A^2}{S_B} \cdot \frac{k_{bw}^2(0)}{k_{ww}(0)}$

The $w_m(\alpha)$ that optimizes S/N for a given pulse shape $b(\alpha)$ is found by exploiting the known property of correlation functions (based on Schwartz's inequality)

x $\quad k_{bw}^2(0) \le k_{bb}(0) \cdot k_{ww}(0)$ that is $\frac{k_{bw}^2(0)}{k_{ww}(0)} \le k_{bb}(0)$

where the maximum is achieved with **filter weighting proportional to the signal shape**

$w_m(\alpha) \propto b(\alpha)$ which normalized to unit area is $\boxed{w_m(\alpha) = b(\alpha)}$

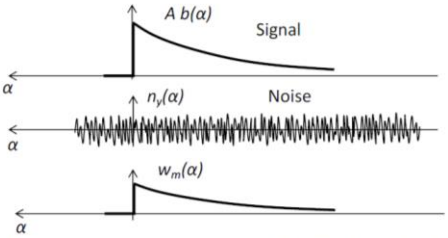
and gives $\max\left[k_{bw}^2(0)\right] = k_{bb}^2(0)$ that is $\max\left[\frac{k_{bw}^2(0)}{k_{ww}(0)}\right] = k_{bb}(0)$

The Schwartz inequality x proves that the maximum of the SNR is when w_m(alpha) = b(alpha), that is when the filtering weighting function shape is proportional to the shape of the input signal.

Every time we choose a matched filter it is almost compulsory to write equations z.

SCHWARTZ INEQUALITY – MATCHED FILTER

For the optimum filter we are weighting the noise and the signal, but since the noise is always the same, better to increase the weight of the filter when the signal is high and reduce it when the signal is small.



The best result in measurements of the amplitude of signal pulses accompanied by **stationary** white noise is obtained with weighting function equal to the signal shape. This conclusion is intuitive: since the noise is uncorrelated, the output noise power is the weighted sum of the noise instantaneous power at all times; since this power is equal at all times, it is convenient to give higher weight when the signal is higher.

The filter with weighting function $w_m(\alpha)$ matched to the signal shape $b(\alpha)$

$$\boxed{w_m(\alpha) \propto b(\alpha)}$$

is indeed called **MATCHED FILTER**

The matched filter is so called because the shape of the filter is the same of the signal, i.e. is the optimum filter.

Optimum SNR for WN

Sb is the bilateral spectral density. k_bb(0) is the autocorrelation in 0 of the signal.

The optimum S/N provided by the Matched Filter is

$$\boxed{\left(\frac{S}{N}\right)^2_{opt} = \frac{A^2}{S_B} \cdot k_{bb}(0) = \frac{A^2}{S_B} \cdot \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha}$$

recalling that the energy E_y of the signal $A\, b(t)$ is

$$E_y = A^2 \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha = A^2 \int_{-\infty}^{\infty} B^2(f)\, df$$

we see that $(S/N)^2_{opt}$ is simply

$$\boxed{\left(\frac{S}{N}\right)^2_{opt} = \frac{E_y}{S_B} = \frac{\text{signal energy}}{\text{noise power density (bilateral)}}}$$

The first formula can be rewritten considering the energy definition.

Starting from these formulas, an interesting point would be to understand which is the minimum amplitude of the signal we can detect.

So we want to revert the formula to get the amplitude. However, at this point we need to define a target SNR, otherwise we cannot reverse the formula. From theory, 1 is the minimum SNR to have a minimum detectable signal. In the real world, with a SNR = 1 we cannot distinguish the signal, at least we need a SNR = 3.

Then we solve the equation for SNR = 1 (from theory) and we get the Amin.

By considering explicitly the energy of the normalized signal *b(t)*

$$E_b = \int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha = \int_{-\infty}^{\infty} B^2(f)\, df$$

we set in evidence the signal amplitude *A* (the signal area)

$$\boxed{\left(\frac{S}{N}\right)^2_{opt} = A^2 \cdot \frac{E_b}{S_B}}$$

The minimum measurable amplitude A_{min} is defined as the amplitude that gives $(S/N)_{opt} = 1$, therefore it is

$$\boxed{A_{min} = \frac{\sqrt{S_B}}{\sqrt{E_b}} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} b^2(\alpha)\, d\alpha}} = \frac{\sqrt{S_B}}{\sqrt{\int_{-\infty}^{\infty} B^2(f)\, df}}}$$

that is A^2_{min} = (spectral density of noise) / (energy of the normalized signal)

OPTIMUM FILTERING WITH ANY STATIONARY NOISE

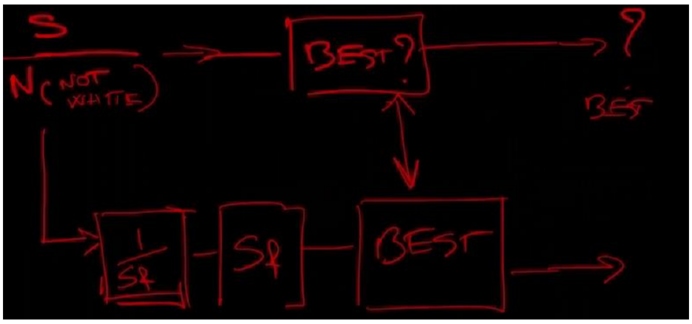
We are considering a stationary noise, not just a WN. Is it possible to apply the same theory? No, because to compute the optimum filter we used an approach that worked with white noise, because writing the noise at the output as Sb*k_ww(0) is correct only if WN.
However, we can get the same result changing something in the filter.

For CPF, they are reversable (i.e. we can always go back) and we can always change the order of the filter. Since it is reversable, every time we apply a filter we can always go back, so we are not loosing any original information. The information might be modified but it is the same.

We want to obtain a situation where SNR in input, we apply the optimum filter and we get SNR_best at the output. This works only if the noise is white.
The problem is that in this case the noise is not white, so we cannot put the matched filter in the middle. But still exists the best filter in this case, we just don't know which is.

I use another approach. We add two blocks, 1/S(f) and S(f), where S(f) is a CP (constant parameter) filter. Since we CPF is reversable, I apply a filter and then I reverse it and apply it again. So after the two filters I have the same information, but also in the middle between the two CP filters, it is just changed, the CPF cannot increase or reduce the amount of information.

The first block can be a filter that makes the noise white, making the spectral density of the noise in the frequency domain flat. So in the middle between the two CPF 1/S(f) and S(f) the noise is white. Hence the S(f) and BEST (that is not the matched) blocks compose the matched filter, because it brings the WN to the optimum SNR in output.



This is not the only way to get the best result, we can define directly the best filter with MATLAB without making the noise white, but the important thing is to get the optimal SNR as a result.

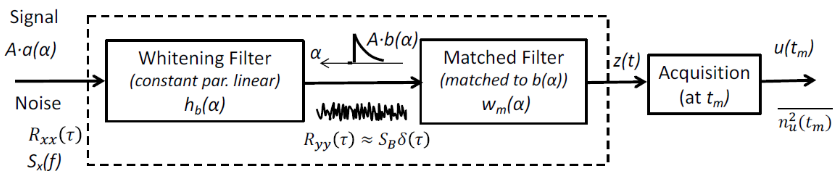
The whitening filter is applied on the noise, but since the filter is in the chain, I'm applying the same fitler also to the signal. So when I design the next matched filter I will design a filter with the same shape of the signal at the output of the whitening filter, not the original signal.

- Optimal filters for measuring the amplitude of pulse signals accompanied by a **any stationary noise** can be obtained by an **extension of the solution for white noise**.
- We can begin by processing the signal and noise with a noise-whitening filter that transforms the noise in white noise by a reversible transformation. In fact:
  a) for a given stationary noise, it is always possible to find a **constant-parameter** linear filter that produces such a result, since it has transfer function $H_b(f)$ such that

$$|H_b(f)|^2 \propto \frac{1}{S_x(f)}$$

  b) the transformations performed by constant-parameter linear filters are reversible.
- **Reversibility is essential**: nothing is lost in the transformation and whatever is done by the whitening filter can be reversed by the following filters. We can then proceed towards the optimum, since we know what to do in the situation at the output of the whitening filter: we have pulse signals accompanied by white stationary noise and we know that a matched filter performs the optimum filtering.
- **In conclusion**, the optimum weighting function for measuring the amplitude of pulses with any stationary noise is obtained as the **overall weighting function of two cascaded filter** stages: a **whitening** filter followed by a **matched** filter.

Optimum filter



- The whitening filter modifies the waveform of the signal, hence the following filter is matched to this modified signal, not to the input signal!
- The subdivision of the optimum filter in whitening filter and matched filter is a useful theoretical approach for analyzing the problem and finding the overall optimal weighting, but it is **NOT THE NECESSARY STRUCTURE** of the the optimum filter.
- In principle, we find the optimal weighting by combining whitening filter and matched filter. In practice, we can implement this optimal weighting with different filter structures employing any kind of linear filter (constant or time-variant parameters; passive or active; etc.)
- The wide liberty in the implementation of the optimum filter is very important, since in many cases it is quite difficult to design the noise-whitening filter and even more difficult to implement it, due to practical limitations of the real components (limited linear dynamic range; noise in the circuit elements; etc.).

- For a given noise with spectrum $S_x(f)$ the **whitening** filter is a **constant-parameter linear** filter that has transfer function $H_b(f)$ such that $|H_b(f)|^2 \propto 1/S_x(f)$ (in time domain: filter autocorrelation function $k_{bb}(\tau)$ such that the convolution with the noise autocorrelation $R_{xx}(\tau)$ produces a δ-like autocorrelation $R_{yy}(\tau) \propto \delta(\tau)$)
- The action of the whitening filter is more evident in cases where the actual noise results from white noise filtered by some circuit. For example, consider the Johnson noise of a resistor passed through an amplifier with upper band-limit set by a simple pole at low frequency. The whitening filter simply reverts the filtering by the amplifier with a transformation that cancels the low-pass pole.

OPTIMUM FILTERING FOR HIGH IMPEDANCE SENSORS

- High impedance sensors and low-noise preamplifiers
- Noise whitening filter
- Matched filter
- Optimum filtering for measuring the charge of pulse signals
- Optimum Filtering with Finite Readout Time
- Practical approximations of the optimum filtering

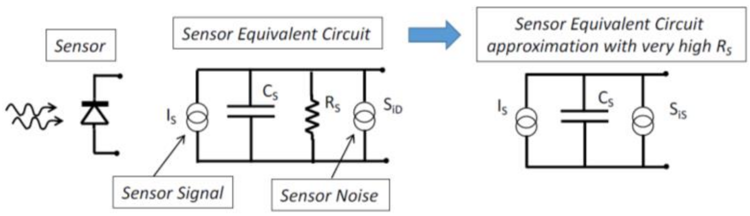
HIGH IMPEDANCE SENSORS AND LOW NOISE AMPLIFIERS

Let's suppose to have a high impedance sensor.

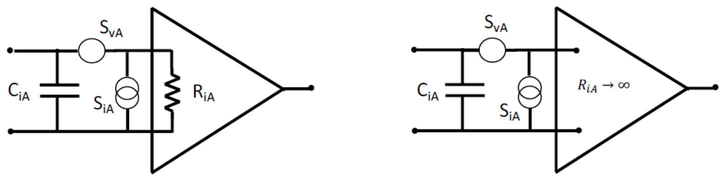
High impedance sensor

It is a current generator for both the signal and the noise, and since it is high impedance we also have a capacitor. The sensor equivalent circuit has two generators, one for the signal and one for the noise, and a parallel capacitor.

- Let us consider sensors that are seen by the circuits connected to their terminals as generators of current signals with high internal impedance (typically a small capacitance C_S with a high resistance R_S in parallel)
- **Typical examples are**: p-i-n junction photodiodes and other photodetectors (CCDs, vacuum tube photodiodes, etc.); piezoelectric Force Sensors in quartz or other piezoelectric ceramic materials
- They have internal noise sources (e.g. shot current noise of a junction reverse current) modeled by a current noise generator in parallel to the signal generator



Low noise amplifier



Preamplifier equivalent circuit → *Approximation with very high R_{iA}*

- R_{iA} = **true physical resistance** between the input terminals (NOT the dynamic input resistance modified by the feedback in the amplifier; e.g. not the low dynamic resistance of the virtual ground of an operational amplifier)
- Besides shot noise of bias currents, the S_{iA} includes Johnson resistor noise of R_{iA}
  $$S_{iR} = \frac{4kT}{R_i}$$
- The current noise directly faces the sensor current signal I_S
  if R_{iA} is small the S_{iR} is overwhelming (e.g. with $R_{iA} = 50\,\Omega$ it is $\sqrt{S_{iR}} \approx 18\,pA/\sqrt{Hz}$)
  and other components of $\sqrt{S_{iA}}$ are much lower (about $1\,pA/\sqrt{Hz}$ or lower)
- Conclusion: for **low-noise** operation of **high-impedance sensors**, it is **mandatory to employ a preamplifier with high input resistance R_{iA}**

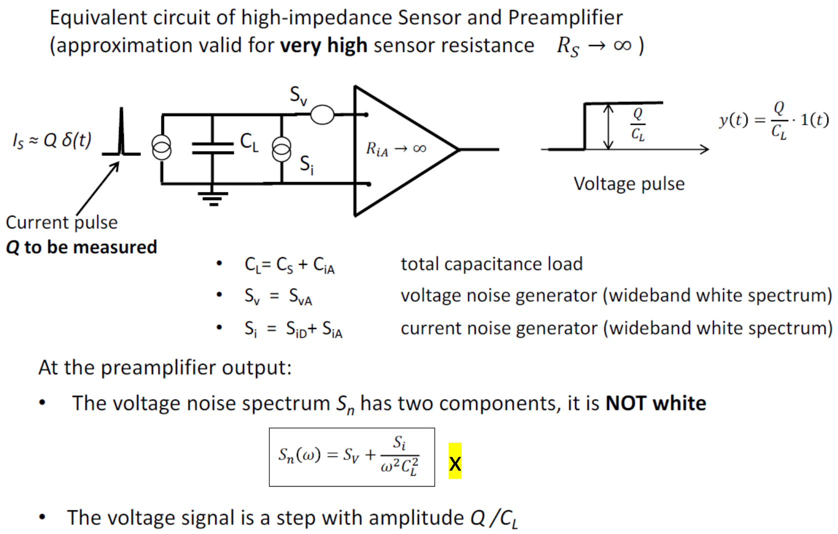
The preamp is used to read the current form the sensor. The input resistance of the preamp is the true physical resistance, because noise is associated with physical resistances.

If it is big we can remove the input resistance, and this is done on purpose because we have thermal noise associated to the resistance, which in terms of current noise depends on 1/R. This noise current generator has to be placed in parallel to the preamp noise generators, which are in parallel with the signal. So if I reduce the resistance I increase the noise in parallel to the sensor.

If for instance we use 50 Ohm and we compute the spectral noise, we get 18 pA/sqrt(Hz). Since the noise of the preamp is typically smaller, we are adding a lot of noise. We considered 50 Ohms because it's the impedance of a transmission line.
The input resistance Ria to be removed works if the preamp system is close to the sensor, otherwise we need to use transmission lines and the 50 Ohm resistance has to be considered.

The input signal can be modelled as a delta, because even if it is an exponential decay time, sometimes the tau is so small that it can be modelled as a delta if the signal is fast. E.g. for a pn junction a delta is a very fast laser pulse that hits a photodiode, which gives us a pulse of current.

Si is the current generator of the preamplifier plus the current generator of the sensor plus the one of the resistance (that has however been removed).



The result is that if we look at the voltage noise, it is not white. The fact that the noise is not white is due to the capacitor Cl of the sensor, because integrating a current over a capacitor gives us a spectrum that is no more flat. A delta of charge in a capacitor gives us a step.

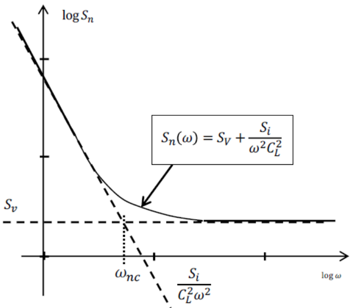
The plot of the result x is in the next image. Where the 1/f noise is equal to the white noise we have the corner frequency.
The tau associated to the noise corner is y. Accordingly, we can also define the noise corner resistance.

Crossing of the component defines
ω_{nc} Noise-Corner angular frequency

$$S_v = \frac{S_i}{C_L^2 \omega_{nc}^2} \Rightarrow \boxed{\omega_{nc} = \frac{\sqrt{S_i}}{C_L \sqrt{S_v}}}$$

$T_{nc} = 1/\omega_{nc}$ Noise-Corner time constant

y $\boxed{T_{nc} = \frac{1}{\omega_{nc}} = \frac{\sqrt{S_v}}{\sqrt{S_i}} C_L}$



T_{nc} and ω_{nc} are fundamental parameters of the optimum filter: we will see that T_{nc} rules the duration of the filter weighting and ω_{nc} the filter bandlimit

We define the **Noise Corner resistance** $\boxed{R_{nc} = \frac{\sqrt{S_v}}{\sqrt{S_i}}}$ so that $\boxed{T_{nc} = R_{nc} C_L}$

- with $\sqrt{S_v}$ a few $nV/\sqrt{Hz}$ and $\sqrt{S_i}$ ranging from a few 0,1 to 0,01 $pA/\sqrt{Hz}$
  R_{nc} ranges from tens to hundreds of kOhms
- with C_L from 0,1 pF to a few pF
  T_{nc} ranges from a few nanoseconds to some hundreds of nanoseconds

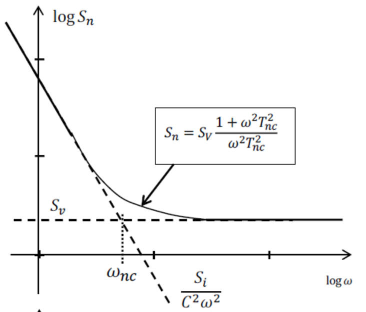
NOISE WHITENING FILTER

The spectral density from which we start is Sn(w), and it has one pole in the origin and a zero. To obtain a flat behaviour I need a filter with a zero in the origin and a pole later.

The noise spectrum has

- a pole at $\omega_p = 0$
- a zero at $\omega_z = \omega_{nc} = 1/T_{nc}$

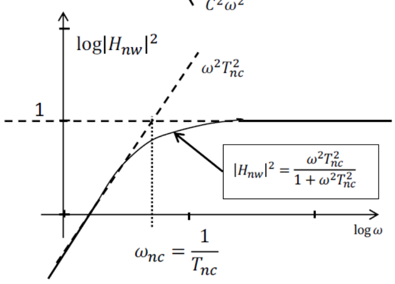
$$S_n(\omega) = S_v\left(1 + \frac{S_i}{\omega^2 S_v C_L^2}\right) = S_v\left(1 + \frac{1}{\omega^2 T_{nc}^2}\right) = S_v \frac{1 + \omega^2 T_{nc}^2}{\omega^2 T_{nc}^2}$$



The **noise whitening filter H_{nw}** must

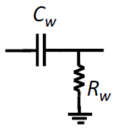
- cancel the pole with a zero at $\omega = 0$
- cancel the zero with a pole at $\omega = \omega_{nc} = 1/T_{nc}$

$$\boxed{|H_{nw}(\omega)|^2 = \frac{\omega^2 T_{nc}^2}{1 + \omega^2 T_{nc}^2}}$$



It is a simple high-pass filter

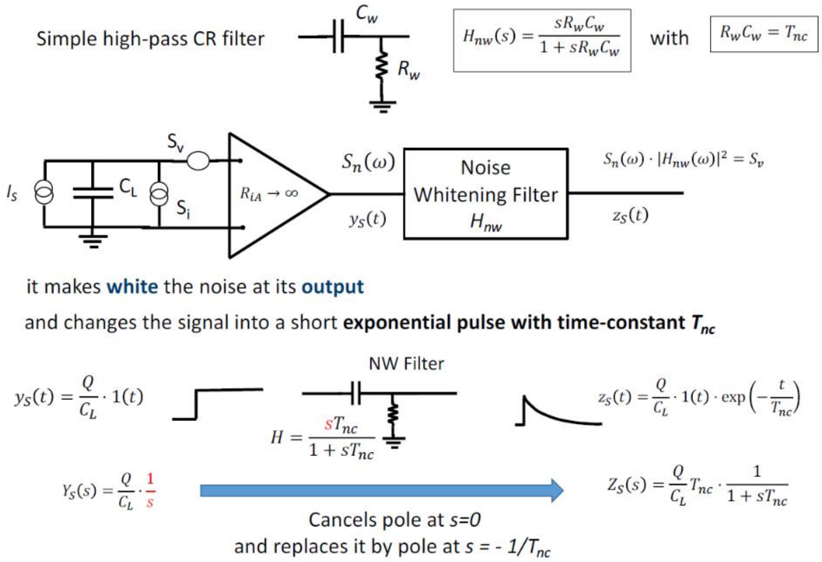
$$H_{nw}(\omega) = \frac{j\omega R_w C_w}{1 + j\omega R_w C_w}$$



$\boxed{\text{with } R_w C_w = T_{nc}}$

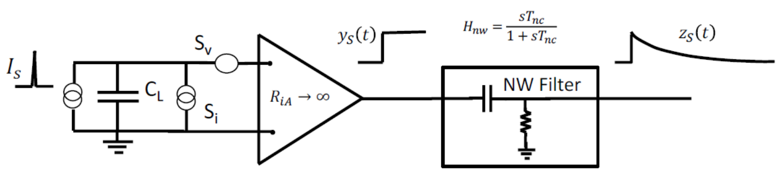
This filter is the HP filter.
Since we are considering the noise, it is important the squared absolute value of Fourier transform of the filter, because we are whitening (1/f)^2, otherwise we would not whiten the just the 1/f noise.

Whitening result



I applied the whitening filter, and we also have to consider the signal. The step in input to the whitening filter gives us an exponential decay time.
Now the noise is white and the signal is the exponential decay time, after the whitening filter. At this point I have to apply the matched filter (filter whose weighting function has the same shape of the signal), and I don't have to apply the RC filter, because the RC has the exponential decay time as a delta response, not as a weighting function, that is the delta response flipped.
Furthermore, the matched filter with this exponential decay shape doesn't exists because it would require an integration for an infinite time.

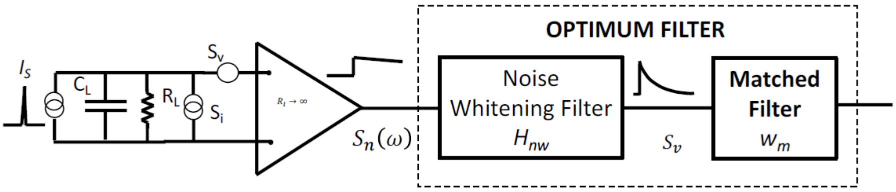
Signal in output to the whitening filter



Input (current)	Preamp Output (voltage)	NW Filter Output (voltage)
δ – pulse	Step pulse $\frac{Q}{C_L}$	Exponential pulse $\frac{Q}{C_L}$ T_{nc}
$I_s(t) = Q \cdot \delta(t)$	$y_S(t) = \frac{Q}{C_L} \cdot 1(t)$	$z_S(t) = \frac{Q}{C_L} \cdot 1(t) \cdot \exp\left(-\frac{t}{T_{nc}}\right)$
$I_s(s) = Q$	$Y_S(s) = \frac{Q}{C_L} \cdot \frac{1}{s}$	$Z_S(s) = \frac{QT_{nc}}{C_L} \frac{1}{1+sT_{nc}}$

The **matched** filter has to be tailored to the signal at the whitening filter output, i.e. it must have **weighting function exponential with time constant T_{nc}**

MATCHED FILTER



In the case with finite load resistance R_L the whitening filter is different but the output signal produced is the same as with $R_L \to \infty$

$$z_S(t) = \frac{Q}{C_L} \cdot 1(t) \cdot \exp\left(-\frac{t}{T_{nc}}\right)$$

Therefore, the matched filter is the same in the two cases

$$\boxed{w_M = 1(t) \cdot \frac{1}{T_{nc}} \exp\left(-\frac{t}{T_{nc}}\right)}$$

and gives the same result

$$\eta_o^2 = \left(\frac{S}{N}\right)^2_{opt} = \frac{\left[\int_{-\infty}^{\infty} z_S(\alpha)\, w_m(\alpha) d\alpha\right]^2}{S_v \cdot \int_{-\infty}^{\infty} w_m^2(\alpha) d\alpha} = \frac{Q^2}{C_L^2}\frac{T_{nc}^2}{S_v}\int_{-\infty}^{\infty} w_m^2(\alpha) d\alpha = \frac{Q^2}{C_L^2}\frac{1}{2}\frac{T_{nc}}{S_v}$$

The red one is the weighting function. As for the signal to noise ratio, we write its definition (integral) for both signal and noise.

OPTIMUM FILTERING

At the output of the optimum filter (i.e. of the matched filter) we have

Signal $$s_o = \int_0^{\infty} z_s(\alpha)\, w_m(\alpha) d\alpha = \frac{Q}{C_L}\frac{1}{T_{nc}}\int_0^{\infty} \exp\left(-\frac{2\alpha}{T_{nc}}\right) d\alpha = \frac{1}{2}\frac{Q}{C_L}$$

Noise $$\sqrt{\overline{n_o^2}} = \sqrt{S_v} \cdot \sqrt{k_{ww}(0)} = \sqrt{S_v}\frac{1}{T_{nc}}\sqrt{\int_0^{\infty} \exp\left(-\frac{2\alpha}{T_{nc}}\right) d\alpha} = \sqrt{\frac{S_v}{2T_{nc}}}$$

S/N $$\boxed{\eta_o = \frac{s_o}{\sqrt{\overline{n_o^2}}} = \frac{1}{2}\frac{Q}{C_L}\sqrt{\frac{2T_{nc}}{S_v}}}$$

- The optimum filter theory specifies what is the best S/N physically obtainable and the kind of filter required for attaining it
- The optimum filter can be implemented in reality, but the filter design turns out to be quite complex
- Furthermore, the optimum filter takes infinite time (after the signal onset) to complete its action, which is not acceptable in practice
- It is possible, however, to consider and evaluate fairly simple filters that approximate the optimum filter and closely approach its performance

The signal in output is the prduct of the weighting function and the singal in input, so the expoenntial decay time squared.
As for the noise of the matched filter, it is 1/(2Tnc), because the weighting function is the exponential decay time, that is not the one of the RC, but **from the noise standpoint having a weighting function or the flipped one is exactly the same**. Since the autocorrelation of the RC is a double exponential with value in 0 of 1/(2Tf), for the matched filter in this case it is the same shape and value in 0.

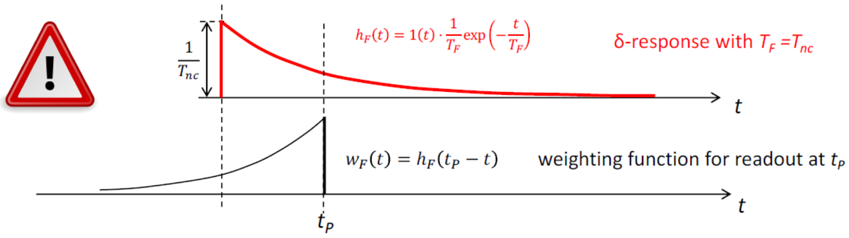
The problem is that this filter doesn't exists. Firstly because we are not able to develop this shape form an analog point of view, and moreover it is an anticausal filter that lasts for infinite time. To solve the problem of anticausality we change the t_m position with a delay line, putting it in the end of the exponential curve and not in correspondence of the peak. The thing is that we are creating an approximation in this way because we are cutting part of the weighting function.

PRACTICAL APPROXIMATIONS OF THE OPTIMUM FILTERING

RC integrator approximation of the matched filter

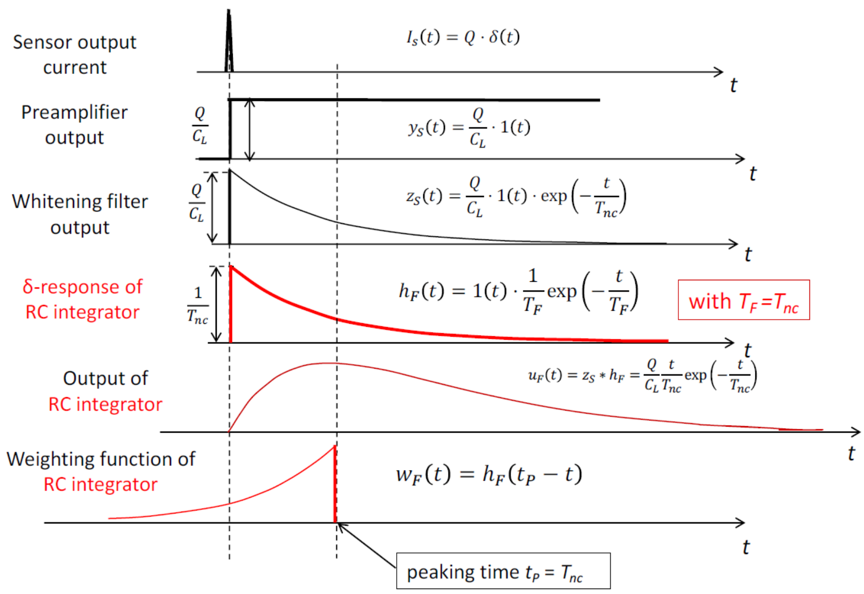
In the image we have the delta response of the RC filter, whose weighting function is not the red one, but the red one flipped and shifted in time.

- The whitening filter is simple and easily and exactly implemented. For completing the optimum filter it is sufficient to find out how to approximate the matched filter.
- The features of the matched filter weighting function observed in time and in frequency point out that it is a low-pass filter
- A simple **RC integrator** (single-pole low-pass filter) can be an **approximation** of the matched filter. With $RC = T_{nc}$ its δ-response $h_F(t)$ is identical to the weighting function $w_M(t)$ of the matched filter. The **RC weighting** $w_F(t)$ has the same shape as $w_M(t)$ of the matched filter, but it's not fully correct because it is **reversed in time!**



- **Noise** filtering is **equal to the matched filter**, since it is unaffected by time-inversion; the output is white noise with band-limit set by a simple pole with time-constant T_F .
- **Signal** filtering is **different** from the matched filter, since it is modified by time-inversion

Let's try to do this error anyway, using an RC instead of the matched filter.



We tune the delta response of the RC with the same tau of the output of the whitening filter to have the best possible RC. It is a CPF, so if I apply it I need to make the convolution of the delta response with the whitening output.

Once we have the output of the RC integrator, I have to take t_m, which is in correspondence of the maximum of the output curve. The maximum is exactly at one tau (at Tnc). Since here I have the maximum, I don't have to perform the convolution but just apply the weighting function at t_m and multiplying it with the signal in output of the whitening filter.

Now to compute the SNR we cannot use the formula of the optimum filter, but the general formula.

Since we calculated the optimum filter and the RC has the same tau = Tnc, the noise is again 1/(2Tnc) for the noise computation, we have just to compute the noise.

The RC output signal waveform is $u_F(t) = \frac{Q}{C_L}\frac{t}{T_{nc}}\exp\left(-\frac{t}{T_{nc}}\right)$

Signal peak value (at $t = T_{nc}$) $s_F = u_F(T_{nc}) = \frac{1}{e}\frac{Q}{C_L}$

Noise $\sqrt{\overline{n_F^2}} = \sqrt{S_v} \cdot \sqrt{k_{hh}(0)} = \sqrt{\frac{S_v}{2T_{nc}}}$

S/N $\eta_F = \frac{s_F}{\sqrt{\overline{n_F^2}}} = \frac{1}{e}\frac{Q}{C_L}\sqrt{\frac{2T_{nc}}{S_v}}$

Comparing the RC approximation with the ideal optimum filter system we see that

$s_F = \frac{2}{e}s_o \approx 0{,}736 \cdot s_o$ the signal is lower

$\sqrt{\overline{n_F^2}} = \sqrt{\overline{n_o^2}}$ the noise is equal

$\eta_F = \frac{2}{e}\eta_o \approx 0{,}736 \cdot \eta_o$ the S/N is lower

the performance of the filter system with RC approximation of matched filter is about 27% worse than the absolute optimum.
Note that the loss is due to bad exploitation of the signal

If we compare the obtained SNR with the one of the matched filter (or we just compare the signals, since the noise is the same), the SNR is 73% lower than the previous case.

1/f NOISE

It is an issue in several applications. Simply making a measurement with the 1/f noise present it's impossible.

1/f NOISE FEATURES

1/f means that the spectral density of the noise is 1/f, even if in the real world it is not exactly 1/f.
From the power point of view it becomes larger and larger. Bipolar transistors have a low 1/f noise, MOSFETs have a strong impact due to the 1/f noise.

However, 1/f is not just related to electronics, it is an issue in several application because its origin is a process that is common also outside the electronics.
The main difference between **1/f** and white noise is that **samples are strongly correlated even at a long time distance**, while for white noise two samples are always uncorrelated regardless the distance between these two samples.

Random fluctuations with power spectral density

$$S(f) \propto \frac{1}{|f|}$$

- first reported in 1925 as «flicker noise» in electronic vacuum tubes
- ubiquitous, observed in all electronic devices
- with very **different intensity in different devices**:
  very strong in MOSFETs; moderate in Bipolar Transistors BJTs;
  moderate in carbon resistors; ultra-weak in metal-film resistors; etc.
- observed in many cases also **outside electronics:**
  cell membrane potential; insulin level in diabetic blood; brownian motion;
  solar activity; intensity of white dwarf stars; ocean current flux;
  frequency of atomic clocks; ... and many others

- Basic **distinction** between 1/f and white noise:
  time span of interdependence between samples
  for **white** noise: samples are **uncorrelated even at short** time distance
  for **1/f** noise: samples are strongly **correlated even at long** time distance

As said previously, 1/f is to the power of alpha in reality, but it really doesn't matter at all if we just use 1/f, it is still a good approximation.
With 1/f^2 it's easy to whiten the noise, while if f is to the power of 1 it is very difficult to whiten it.

The 1/f comes from the release of carriers trapped in the channel of a mosfet but with different times. In general, when we have events happening at different times, their sum is the 1/f noise.

- The real observed power density at low frequency is often not exactly $\propto \frac{1}{f}$ but rather $\propto \frac{1}{|f|^{\alpha}}$ with α close to unity, i.e. $0{,}8 < \alpha < 1{,}2$

  anyway the behavior of such noise is **well approximated by *1/f*** density

- 1/f noise arises from physical processes that generate a **random superposition** of elementary pulses with **random pulse duration** ranging from **very short to very long**.

E.g. in MOSFETs 1/f noise arises because:

a) carriers traveling in the conduction channel are randomly captured by local trap levels in the oxide, stop traveling and stop contributing to the current
b) trapped carriers are later released by the level with a random delay
c) the level lifetime (=mean delay) strongly depends on how far-off is from the silicon surface (= from the conduction channel) is the level in the oxide
d) trap levels are distributed from very near to very far from silicon, lifetimes are correspondingly distributed from very short to very long

EXAMPLE

1/f NOISE SPECS

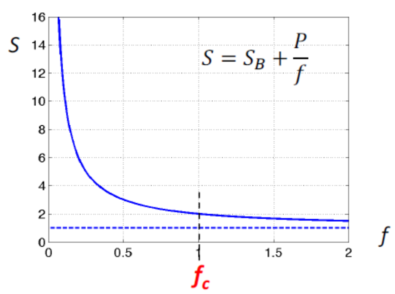
P is the intensity, but it is a constant that is not known generally. Normally in a datasheet we have the frequency corner of the integral of 1/f. To make the power we have to make the integral from 0 to +inf of P/f (we are using the unilateral, otherwise from -inf but with the modulus). Here we have the first problem, because normally we have also the white noise in our system.

Since we have the white noise we can define the frequency corner, that is the point where the 1/f noise crosses the white noise. Clearly, it is better to have a small frequency corner so that I can increase the portion of the spectrum where the noise is just white. But there is also another reason; in fact, we have less noise if we decrease the frequency corner. <u>Every time we shift the frequency corner to high frequencies we are increasing the noise</u>.

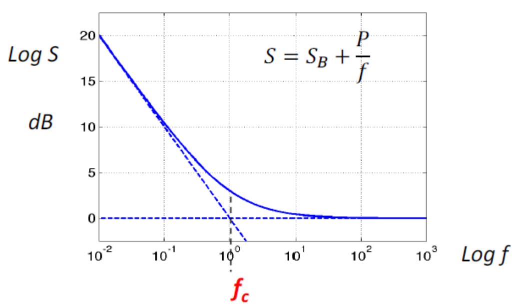
- Spectral density $S_f(f) = \frac{P}{f}$ noise power $\overline{n_f^2} = \int_0^{+\infty} \frac{P}{f}\,df$ (with unilateral S_f)

- circuits and devices have both 1/f noise S_f and white noise S_B

Linear diagram



Bode diagram



- S_f is specified in relative terms referred to the white noise S_B by specifing the «corner frequency» f_c at which $S_f = S_B$

Frequency corner

Since P is related to the power of the 1/f noise, as soon as we increase fc we increase the power related to the 1/f noise.

- The **1/f noise corner frequency** f_c is defined by

$$\frac{P}{|f_c|} = S_B \qquad \text{hence} \qquad P = S_B f_c$$

NB: the **higher** is frequency f_c
the **stronger** is the role of 1/f noise
and for a given S_B , the higher is the intensity P

Typical values for low-noise voltage amplifiers :

- S_B a few 10^{-18} V^2/Hz → $\sqrt{S_B}$ a few $\frac{nV}{\sqrt{Hz}}$
- f_c 10Hz to 10kHz, that is
- P a few 10^{-17} to a few 10^{-14} V^2 → $\sqrt{P}$ from a few nV to a few 100 nV

White noise is in the range of nV/sqrt(Hz), so the fc goes typically from 10 Hz to 10 kHz. For standard applications, this is a big value, so we cannot neglect it.
Sb is the spectral density of the WN, but if we are using the unilateral spectral density for the 1/f noise, we have to use it also for the WN.

1/f BAND LIMITS AND POWER

The ideal 1/f noise spectrum runs from f = 0 to f → ∞ and has divergent power $\overline{n_f^2}$ → ∞
(recall that also the ideal white spectrum has $\overline{n_B^2}$ → ∞)

$$\overline{n_f^2} = \int_0^\infty \frac{P}{f}\, df \to \infty$$

A **real** 1/f noise spectrum has **span limited** at both ends and is **not divergent**.

If there is **wide spacing** between the high-frequency and low-frequency limitations they can be **approximated by sharp cutoff** at low frequency f_i and high frequency $f_s >> f_i$ and the noise power can be evaluated as *

x $$\overline{n_f^2} \approx \int_{f_i}^{f_s} \frac{P}{f}\, df = P \ln\left(\frac{f_s}{f_i}\right) = S_B f_c \ln\left(\frac{f_s}{f_i}\right)$$

The actual 1/f bandlimits f_s and/or f_i of given filter types will be illustrated later.

---

* Beware !
ONLY if $f_s >> f_i$ the sharp cutoff gives a **GOOD APPROXIMATION** of the noise power !

Theoretically, 1/f at 0Hz is infinite, while goes to -inf at infinite frequency, so it is not limited. We need something that goes to infinite in both the directions. If I apply no filter, the integral of 1/f is hence infinite.

One of the reasons why we can integrate the 1/f noise and not getting infinite is that we are limited by the instrumentation, so we cannot read up to an infinite frequency. So we have no problem in managing the HF, but also the LF is not a problem because 0 frequency would mean that we have to observe our signal for an infinite amount of time, but this is not actually feasible.

So if we have both a limitation at LF and HF (fs and fi), and these two limitations are a sharp cut off and far from each other, we can try to calculate the 1/f noise. Hence I integrate between fs and fi.

Formula x gives us some information. In fact, power of the noise depends on Sb, fc and the ln of the ratio between fs and fi. This is a good thing because WN depends linearly on the frequency Sb*(fs-fi), and the logarithmic dependance is weaker. Furthermore, the 1/f noise depends on the ration between fs/fi, while the WN on the difference between, fs and fi.

In cases with **widely spaced bandlimits** $f_s >> f_i$ the 1/f noise power $\overline{n_f^2}$ is

$$\overline{n_f^2} \approx \int_{f_i}^{f_s} \frac{P}{f}\, df = S_B f_c \ln\left(\frac{f_s}{f_i}\right)$$

NOTE THAT:

- $\overline{n_f^2}$ is divergent for $f_s \rightarrow \infty$ (like white noise).
  A limit at high frequency is necessary for avoiding divergence, but in real cases a finite limit always exists.
- $\overline{n_f^2}$ is divergent for $f_i \rightarrow 0$ (like random-walk noise $1/f^2$).
  A limit at low frequency is necessary for avoiding divergence, but we will see that in real cases there is always a finite limit
- $\overline{n_f^2}$ depends on the **ratio f_s/f_i** and **NOT the absolute values fs and f_i**

We have to select fs and fi.

1/f noise is slowly divergent for fi →0 and fs → inf, because we have the logarithmic of the ratio.

$$\overline{n_f^2} \approx \int_{f_i}^{f_s} \frac{P}{f}\, df = S_B f_c \ln\left(\frac{f_s}{f_i}\right)$$

Note 1: $\overline{n_f^2}$ is **SLOWLY** divergent for $f_i \rightarrow 0$ or $f_s \rightarrow \infty$

Logarithmic dependence → $\overline{n_f^2}$ slowly increases with f_s/f_i

e.g : **x 10 multiplication** of f_s/f_i → **+ 2,3 addition** to $\ln(f_s/f_i)$

EXAMPLE: 1/f noise with $\sqrt{P} = \sqrt{S_B f_c} = 100nV$

a) filtered with f_i = 1kHz and f_s = 10kHz (f_s / f_i = **10**)

$$\sqrt{\overline{n_{f,a}^2}} = \sqrt{2{,}3}\,\sqrt{S_B f_c} = 151nV$$

b) filtered with f_i = 1 Hz and f_s = 10MHz (f_s / f_i = **10^7**, i.e. **x 10^6 higher**)

$$\sqrt{\overline{n_{f,b}^2}} = \sqrt{7 \cdot 2{,}3}\,\sqrt{S_B f_c} = 401nV$$ (just **x 2,7 higher**)

The problem is that to decrease the noise we have to make a strong change in the fs/fi ratio, but since the change in the noise value when fs and fi change is almost the same, even if we change fs and fi we don't gain much, we will be always limited by the 1/f noise.

The good news is that it is not necessary to know the exact value of 1/f noise.

$$\overline{n_f^2} \approx \int_{f_i}^{f_s} \frac{P}{f}\, df = S_B f_c \ \ln\left(\frac{f_s}{f_i}\right)$$

Note 2 : reasonably approximate bandlimits are adequate for estimating $\overline{n_f^2}$
it is not necessary to know very precisely f_s and f_i !!

EXAMPLE: for 1/f noise with $\sqrt{P} = \sqrt{S_B f_c} = 100nV$ we estimate

a) with bandlimits f_i = 1kHz and f_s = 10kHz

$$\sqrt{\overline{n_{f,a}^2}} = \sqrt{2{,}3}\,\sqrt{S_B f_c} = 151nV$$

EXAMPLE

b) with bandlimit f_s corrected to $f_{sn} = \frac{\pi}{2} f_s = 15{,}7\ kHz$ (50% higher)

$$\sqrt{\overline{n_{f,b}^2}} = \sqrt{2{,}75}\,\sqrt{S_B f_c} = 166\,nV \quad \text{(just 10 % higher)}$$

In the example we are using the equivalent noise BW for the WN, which makes no sense for the fs of 1/f noise. The error we are making in the 1/f noise is just 10% higher, even if the difference in frequency is 50%.

1/f NOISE FILTERING

$$\overline{n_f^2} = S_B\ f_c \int_0^{\infty} |W(f)|^2 \frac{df}{f} \ = S_B\ f_c \int_{-\infty}^{\infty} |W(f)|^2 d(\ln f)$$

Filtering of *1/f* noise can be better understood by changing variable from *f* to *lnf*
(beware: it's NOT A BODE diagram: the vertical scale is linear !!)

- **1/f** noise: filtered power $\overline{n_f^2}$ ∝ **area of** $|\mathbf{W}|^2$ **plot in logarithmic frequency** scale

which is different from the case of

- **white** noise: filtered power $\overline{n_B^2}$ ∝ **area of** $|\mathbf{W}|^2$ **plot in linear frequency** scale

In both cases the noise power depends mainly on the **frequency span covered** by $|\mathbf{W}|^2$, delimited by upper and lower bounds in frequency. However, the frequency span is **measured differently** :

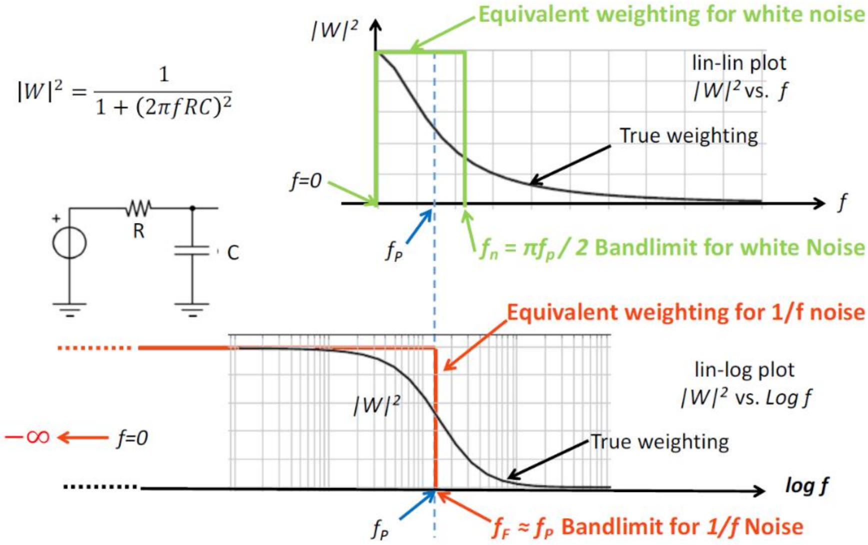
- for **white** noise, by the **difference** of the bounds
- for **1/f** noise, by the **logarithmic difference**, i.e. by the **ratio** of the bounds

Firstly we change the integration variable. Since I'm filtering the noise, the output noise is the integral from 0 to +inf (unilateral) of the spectral density of the noise Sb*fc/f times the absolute value squared of the w_f. With 1/f we will always work in the frequency domain, because to solve this integral in the time domain I would need the autocorrelation of the filter and of the 1/f noise, which is the antifourier transform of the function 1/f, which is not mathematically possible.

The new integration variable is d(ln(f)). 1/f works in the logarithmic scale, while WN in linear scale. The result is that we can use a graphical approach to compute the noise both for the WN and 1/f noise. With WN we plot the w_f in a linear-linear scale and compute the area, while for the 1/f we need a linear-log scale.

- The **band-limits of a filter for white noise** are well visualized in the **linear-linear diagram** of the weighting function $|W(\boldsymbol{f})|^2$:
  the simple equivalent weighting function is rectangular with area and height equal to the true weighting $|W(\boldsymbol{f})|^2$
- The **band-limits of a filter for 1/f noise** are well visualized in the **linear-log diagram** of the weighting function $|W(\ln \boldsymbol{f})|^2$:
  the simple equivalent weighting function is rectangular with area and height equal to the true weighting $|W(\ln \boldsymbol{f})|^2$

Example: RC integrator

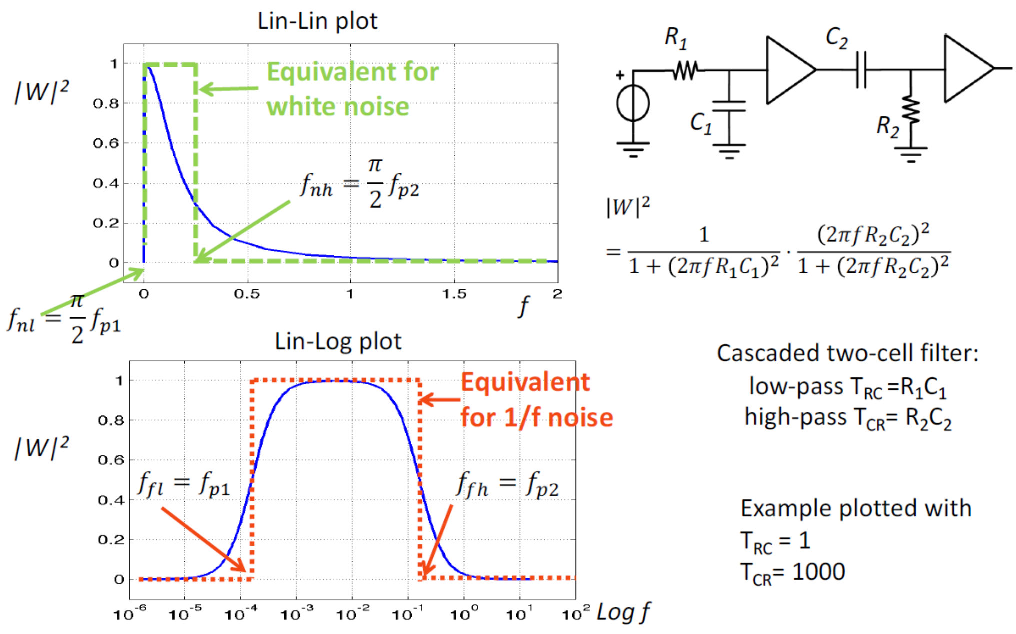


For the WN we have to compute the area of the true w_f (black), that is equivalent to the area of the green rect (that is Sb*fs). For the 1/f we have to plot the w_f of the filter in linear-log scale and then make the integral.
We get the second plot, with the red one that is the approximation. The value of the integral is infinite. It is expected because we have just a LP filter, we are not setting a limitation at LF.

Let's include a limitation also at LF using a HP filter. For the white noise the computation, compared to the previous one, it is the same, because if fi is much smaller than fs, we are basically integrating the same area (fs – fi remains almost equal to fs).

As for the 1/f noise, now we have both a cutoff at LF and HF. The area is the one of the blue curve.
Now, if I plot on the same graph two different filters, for the white noise, comparing the area of the square modulus of the w_f (|W(f)^2|), the smaller the area the better the filter, and so to compare the filters we could theoretically compare the areas. The same is valid for the 1/f noise in the linear-log scale, without making the actual computations.

This is important because we want to compare the filter developed in the next and the CR.



INTRINSIC HIGH PASS FILTERING VS CORRELATED DOUBLE SAMPLING (CDS)

Every time we make a measurement we will use this filter, but this filter also doubles the noise.

- In all real cases, even with DC coupled electronics:

  weighting is inherently NOT extended down to zero frequency,

  because an intrinsic high-pass filtering is present in any real operation.

- The intrinsic filtering action arises because:

  a) operation is **started at some time before** the acquisition of the measure and

  b) operation is **started from zero** value

- EXAMPLE: measurement of amplitude of the output signal of a DC amplifier.
  Zero-setting is mandatory: the baseline voltage is preliminarly adjusted to zero, or it is measured, recorded and then subtracted from the measured signal.
  It may be done a long time before the signal measurements (e.g. when the amplifier is switched on) or repeated before each measurement; it may be done manually or automated, but it must be done anyway.
  Zero-setting produces a high-pass filtering: let us analyze why and how

The real idea is that we don't make a measurement for an infinite time. For instance, if we have to do a measurement, we turn it on, we calibrate it and then we perform the measurement. This is what we have to do to measure the 1/f noise.

The zero setting is mandatory. Every time we make the measurement this is the first thing to do; we can make a measurement with 1/f noise because someone made the zero setting. Now we have to understand the zero setting.

From a signal perspective, what is a zero? What is its effect on 1/f and WN?
The effect on the 1/f noise is that we are doubling the noise. How come?

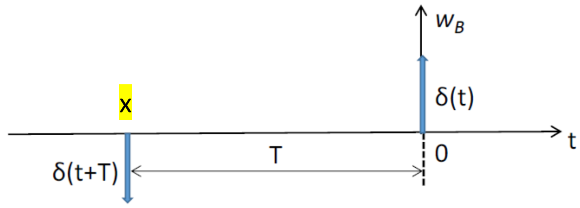
ZERO SETTING BY CDS

Let's start with a measurement; a measurement is a S&H at this moment, I'm sampling the signal, so I have deltas. So at 0 time I have the delta that I use to measure the signal. But before we measure the signal, somewhere in the past, someone, that produced our system, implemented a measurement of the zero, which will be subtracted to the measurement I'm making currently. Since it is a subtraction, it is a delta with the -1 (x). This is the zero setting.

Baseline sample subtracted from signal sample, **both** acquired with **instant sampling**

Time-domain weighting

Y $w_B(t) = \delta(t) - \delta(t+T)$



Frequency-domain weighting

$$W_B(\omega) = F[w_B(t)] = 1 - e^{i\omega T} = 1 - \cos\omega T - i\ \sin\omega T$$

Since: $\cos\omega T = \frac{1}{2}(e^{i\omega T} + e^{-i\omega T})$ $\sin\omega T = \frac{1}{2i}(e^{i\omega T} - e^{-i\omega T})$

For noise $|W_B(\omega)|^2 = [1 - \cos\omega T]^2 + \sin^2\omega T = 2\ [1 - \cos\omega T]$

We can also write z $|W_B(\omega)|^2 = 4\sin^2\left(\frac{\omega T}{2}\right)$ [since it is *(1-cosx) = 2 sin²(x/2)*]

At ωT<< 1 a **low frequency cutoff** is produced

$$|W_B(\omega)|^2 \approx \omega^2 T^2$$ (for x << 1 it is *sinx≈x* and *cosx ≈ 1 – x²/2*)

In the real world, we don't use delta to acquire signals, are we sure we use it for the offset? We will have the same problem, because acquiring the signal or the baseline (offset) is exactly the same.
The fact that the noise is doubled for the WN is in the fact that we are acquiring a signal two times, one for the baseline and one for the actual signal, so we are acquiring the noise twice, because the useful signal is actually sampled only once in 0.

So we have to express the weighting function, that is a delta at t minus it translated in time (y). Then we have to shift to the frequency domain, because the goal is to compare this filter with a HP filter to find a cutoff frequency, and a HP filter is much easier to be treated in the frequency domain.
The Fourier transform of the sum is the sum of the Fourier transform, so we apply the Fourier transform to the delta adding the exponential that gives the shift in time. Then the exponential is rewritten using Euler's expansion.

Then we take the absolute value squared of the w_f. Expression z is useful because we can approximate it to (wT/2)^2 when wT is really small, with then the factor 4.

Let's now plot the w_f. We want to compare the w_f of the zero with an HP filter, and understand if the zero gives us a cutoff at lower frequencies.
At LF, we have the CR and the w_f very similar from the math point of view, even if RC (tau) and T are two completely different things. Let's plot the w_f with T = RC.

Baseline subtraction with delay T

$$|W_B(\omega)|^2 = 4\sin^2\left(\frac{\omega T}{2}\right)$$

at low-frequency ω << 1/T

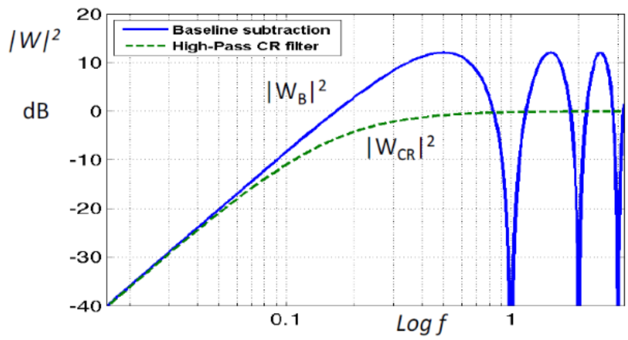
$$|W_B(\omega)|^2 \approx \omega^2 T^2$$

High-Pass CR filter (differentiator)

$$|W_{CR}(\omega)|^2 = \frac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}$$

at low-frequency ω << 1/RC

$$|W_{CR}(\omega)|^2 \approx \omega^2 R^2 C^2$$



BODE DIAGRAM
highlights
the low-freq cutoff

Examples with
equal cutoff T=RC
plotted for T=1

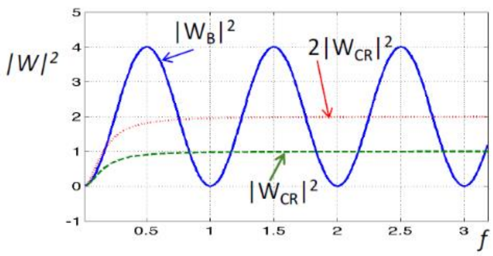
Blue and green curves are different, they are not the same filter, but we are interested in the LF, <u>0Hz</u>, and we notice that <u>the zero setting is equivalent to HP filter. If so, at lower frequency the zero setting is introducing a cutoff in frequency</u>, that is equivalent to introducing a CR with tau = T.

The problem is that we have to understand if T can be chosen or not, and if yes, which is the value to be selected.

CDS VS CR HP FILTER – WHITE NOISE

The first comparison is with WN, then we will do 1/f.
To compare two different filters for the WN, we need to plot the w_f in the lin-lin scale. In green we have the CR, in blue the CDS.
Let's plot the green times a factor two (red), that is the average value of the sinusoidal waveform that goes from 0 to 4. I'm interested in the average value because for the WN we want the integral, i.e. the area, and the integral of a sinusoidal is its mean value multiplied by the sinusoidal between -1 and 1, so it is the mean value.

The CDS has an area of 2, as for the CR, the area is almost 1, not exactly 1 because I loose something at 0Hz. Why if the noise is exactly doubled, here I have almost a factor 2 between the areas of the w_f?
The difference is that we were comparing the sampling without a filtering when saying it is exactly 2, while here we are comparing the CDS with the CR.



LIN –LIN DIAGRAM
white noise power ∝ area of $|W|^2$

NB: examples with equal cutoff T=RC plotted for T=1

White noise $\overline{n_B^2}$ limited also by a low-pass f_s , but with f_s >> $1/T$ and f_s >> $1/RC$

$$\overline{n_B^2} = S_B \int_0^{fs} |W(f)|^2 \, df$$

CDS: $|W_B|^2$ oscillates around 2; its area is **exactly the same** as for a constant $|W_B|^2 = 2$

CR: $|W_{CR}|^2$ has a cutoff at low frequency $f < f_i$ =1/4RC; at higher frequency it is $|W_{CR}|^2 \approx 1$

Therefore, for white noise the output power of the CDS is double of the unfiltered noise and approximately double of the filtered output of the CR (actually even more than double !)

The output power of the CDS is double of the unfiltered noise. If we use the CR we are filtering.

$$\overline{n_B^2} = S_B \int_0^{fs} |W(f)|^2 \, df$$

With Baseline sampling & subtraction	With CR high-pass filter
$\overline{n_B^2} = S_B \int_0^{fs} 2 \cdot [1 - \cos \omega T] \; df$ that is $\overline{n_B^2} = 2 \; S_B f_S$	$\overline{n_B^2} = S_B (f_S - f_i)$ and since $f_s >> f_i$ $\overline{n_B^2} \approx S_B f_S$

Double White noise power, as intuitive because:

1. white noise is acquired twice, in the baseline sampling and in the signal sampling.
2. The two noise samples are uncorrelated, hence their power is quadratically added.

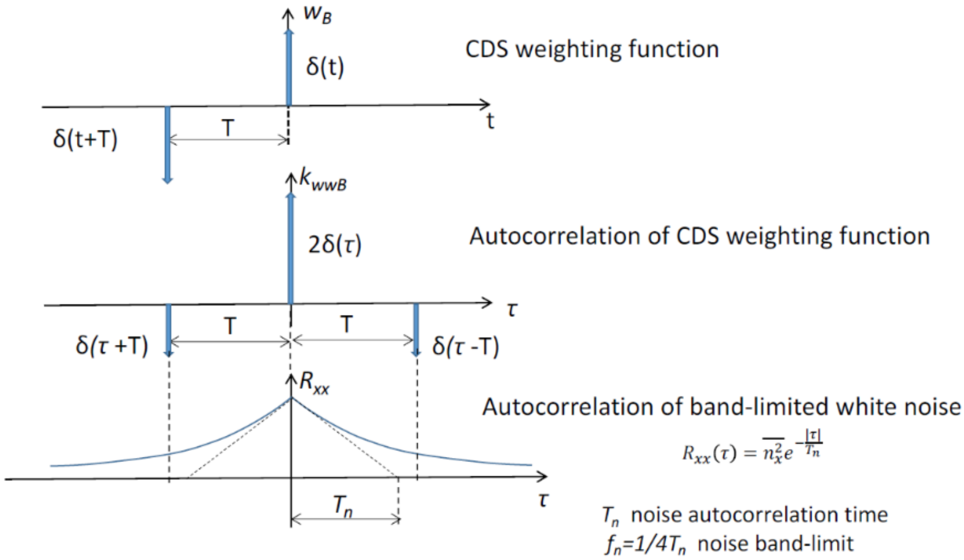
So we are doubling the unfiltered noise and more than doubling with the CR.
T is the delay between when I make the zero and then I perform the measurement. Sometimes T can be chosen and can be small, because we know when we make the measurement. In other cases we don't know when we are making the measurement, so we have to wait and consider the worst case. If T is huge, the tau of the CR is a very low frequency, so we are picking a lot of 1/f noise.

Time domain: filtering band-limited WN by CDS

In the time domain we have to make the integral of the autocorrelation of the filter times the autocorrelation of the noise. The last plot is the autocorrelation of a WN limited by a single pole.

We have just to multiply the second plot and the third one. If the noise was really white, so the BW is much larger than our time scale, the autocorrelation would have been a delta, but I can approximate it with a triangle with Tn very small with respect to my time scale, and T defines the time scale. So it Tn << T, the double exponential goes to much before 0, so we are doubling the noise when we perform the multiplication.

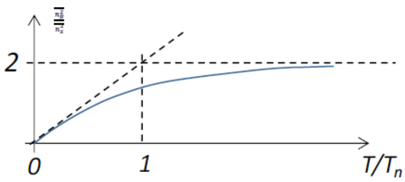
Instead, if the autocorrelation of the noise is very large (not white), the output noise is due the acquisition 2 times in the middle and then one time per side, but if I enlarge the autocorrelation of the noise, in output I get 0 because the side sampling becomes close to 2 times the sampling in the origin.



Extreme cases

$$\overline{n_B^2} = \int_{-\infty}^{\infty} R_{xx}(\tau) k_{wwB}(\tau) d\tau = 2\overline{n_x^2} - R_{xx}(T) - R_{xx}(-T)$$

$$\boxed{\overline{n_B^2} = 2\overline{n_x^2} \cdot \left(1 - e^{-\frac{T}{T_n}}\right)}$$



- Noise with **very short correlation** time (i.e. very high band-limit) is **doubled**:

  if $T_n \ll T$ we have $\overline{n_B^2} \approx 2\overline{n_x^2}$

- Noise with **long correlation time** (i.e. very low band-limit) is **strongly attenuated**:

  if $T_n \gg T$ we have $\overline{n_B^2} \approx \overline{n_x^2} \cdot 2\frac{T}{T_n} \ll \overline{n_x^2}$

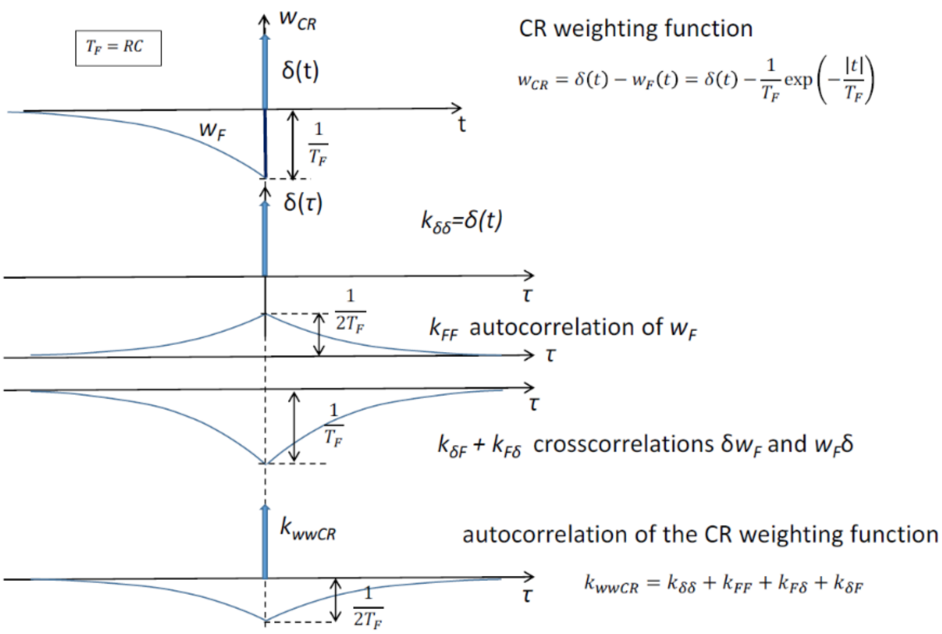
Time-domain analysis clearly shows how with band-limited white noise the output noise power of CDS is double of that of a CR constant-parameter filter with equal cutoff, i.e. with $T_F = RC = T$ Unfiltered noise

Now I have to do the same thing with the CR.

Filtering band-limited WN by CR

Wf is the delta response of the CR flipped. The delta response of the HP filter is 1 – the delta response of the LP filter.

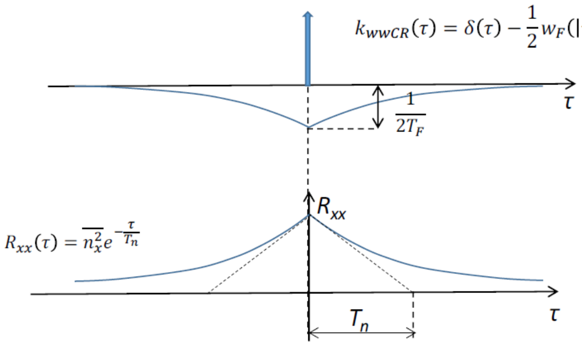
The w_f is the sum of two component, an exponential and a delta. So for the autocorrelation ido the autocorrelation of the delta, of the exponential and the autocorrelation of the cross-product.
The autocorrelation is needed because we are trying to study the 1/f in the time domain for the CDS.



We have to multiply the two plots in the next image. Our goal is to understand what happens when we have a noise with a very short correlation time (with noise) or high correlation time (1/f). If t_m is very small, so ideal white noise because of short correlation time, at the output we have exactly the input. This has to be compared with the correlated double sampling, where we have a factor 2 with WN.
It t_m is very large, the output noise is the input noise attenuated, that is the same situation of the CDS.

So for both CDS and Cr, when the correlation time is long we are attenuating the noise, but when the noise has a short correlation time, in one case CDS we have a doubling of the noise, in the other case just the input noise.

$$k_{wwCR}(\tau) = \delta(\tau) - \frac{1}{2}w_F(|\tau|) = \delta(\tau) - \frac{1}{2T_F}\exp\left(-\frac{|\tau|}{T_F}\right)$$



$$\overline{n_B^2} = \int_{-\infty}^{\infty} R_{xx}(\tau)k_{wwCR}(\tau)d\tau =$$
$$= R_{xx}(0) - 2\int_0^{\infty} R_{xx}(\tau)\frac{1}{2}w_F(|\tau|)d\tau =$$
$$= \overline{n_x^2} - \overline{n_x^2}\int_0^{\infty}\frac{1}{T_F}\exp\left[-\tau\left(\frac{1}{T_n}+\frac{1}{T_F}\right)\right]d\tau$$

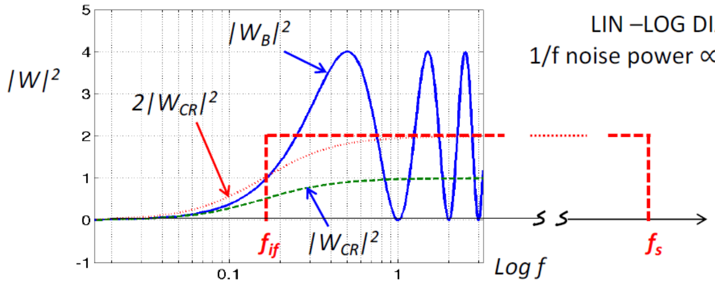
$$\boxed{\overline{n_B^2} = \overline{n_x^2}\left(1 - \frac{T_n}{T_F + T_n}\right) = \overline{n_x^2}\frac{T_F}{T_F + T_n}}$$

- Noise with **very short correlation time** (i.e. very high band-limit) is practically **passed as it is, not doubled** as for CDS: if $T_n \ll T_F$ we have $\overline{n_B^2} \approx \overline{n_x^2}$
- Noise with **long correlation time** (i.e. very low band-limit) is strongly **attenuated at half the level of CDS**: if $T_n \gg T_F$ we have $\overline{n_B^2} \approx \overline{n_x^2} \cdot \frac{T_F}{T_n} \ll \overline{n_x^2}$

So it seems that the CR is better than CDS, because we have no doubling of the noise. We still have a problem, in fact also with a factor 2 I would use a CDS instead of the CR.
With CDS I'm acquiring the signal with a delta, here with a shape that is not a delta, so I'm losing some frequencies, but the real difference, from theory, is that the CR is a filter we are applying on the chain after the signal, so directly on the signal and the noise we have to study the effect of the CR. So if the

signal has some components at LF, with the CR we could damage the signal. The CDS instead is acquired at the zero of the measurement, so we acquire the baseline and then when we acquire the signal we are acquiring all the signal, not with also the filter. With CDS we are just choosing the cutting of the 1/f but not changing the signal.

CDS VS CR HP FILTER: 1/f NOISE



LIN –LOG DIAGRAM
1/f noise power ∝ area of $|W|^2$

Examples with equal cutoff T=RC plotted for T=1

1/f noise power $\overline{n_f^2}$ limited also by a low-pass f_s, but with $f_s >> 1/T$ ($f_s >> 1/RC$)

$$\overline{n_f^2} = \int_0^{fs} |W(f)|^2 \frac{S_B f_C}{f} df = S_B f_C \int_0^{fs} |W(f)|^2 d(\ln f)$$

At low frequency $f<<1/T$ the $|W_B|^2$ and $|W_{CR}|^2$ have the same cutoff (with $T=RC$).
At higher frequency W_{CR} is constant $|W_{CR}|^2 \approx 1$ whereas the $|W_B|^2$ oscillates around a mean value 2, so that :

$$\int_0^{fs} |W_B(f)|^2 d(\ln f) \approx 2 \int_0^{fs} |W_{CR}(f)|^2 d(\ln f)$$

For the 1/f we have to plot the w_f of the CR and CDS (blue) in a linear log scale. The area of the blue line and of the green line (and red dotted line) is the amount of noise we are collecting.
Also for the 1/f, the CDS oscillates around a factor 2 and goes between 0 and 4. At first approximation, also for the 1/f we are doubling the noise. This seems strange because we are using it to remove the 1/f but in reality we are doubling it. The reality is that without CDS, the 1/f is infinite; then we apply CDS and we get a 1/f noise that is doubled with respect to the 1/f that we could get with a CR. Again, it would be better to use the CR because we don't have the doubling of the noise but the problem is that it applies also on the signal.

There's never an optimal solution, we need a tradeoff between not touching the signal and keeping high noise or viceversa.

Therefore

$$\overline{n_{f,\,B}^2} \approx 2\,\overline{n_{f,\,CR}^2}$$

the 1/f noise power output of CDS is approximately double (actually even more than double!) with respect to a CR high-pass with equal cutoff, i.e. with RC=T

For the CR filter it will be shown that the high-pass band-limit for 1/f noise is

$$f_{if} \approx f_p = \frac{1}{2\pi RC} \qquad \text{and} \qquad \overline{n_{f,\,CR}^2} = S_B f_C \ln\left(\frac{f_S}{f_{if}}\right)$$

By comparing the cut-off behavior of CDS and CR, we can conclude that for CDS

$$f_{if} \approx \frac{1}{2\pi T} \qquad \text{and} \qquad \overline{n_{f,\,B}^2} \approx 2 S_B f_C \ln\left(\frac{f_S}{f_{if}}\right)$$

Which is the limit at LF that the CDS sets from the noise point of view? For the CR and 1/f it is the pole (f_ip), while for the CDS is 1/2*pi*T, where T is the distance. So for the noise computation we need to set Sb, fs, fc and f_if. For the lower limit f_if we have to set it as in the previous image depending on having the CR or CDS.

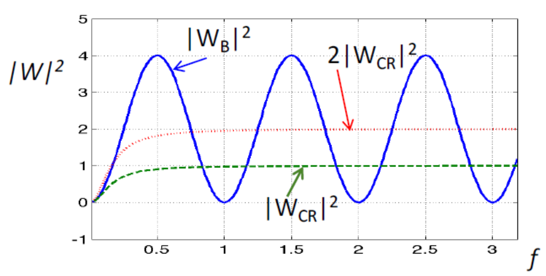
Zero setting by CDS

The doubling of the noise by the CDS is not the only problem, and plus it can be also somehow solved. The real problem is that we don't know the time that passes between the turn on of the instrument and calibration and the measurement. This time is not fixed; to compute the 1/f we have to take the worst case in terms of elapsed time between the calibration and the measurement. And this is not good, having a SNR that changes depending on the time instant on which I make the measurement. I want to remove this dependance.

- Zero-setting by **correlated double sampling (CDS)** produces a high-pass filtering action that limits the power of 1/f noise.
- The **interval T between zero setting and measure in most real cases is quite long** (from a few seconds to several minutes) so that the high-pass **band-limit f_{if}** is quite **low.** This is a main drawback: the filtering is **not very effective** since the 1/f noise power is limited just to a moderately low level, which may be higher than that of white noise.
- Further drawback: **with respect to CR** high-pass filter with equal bandlimit f_{if} the output noise power is **approximately double** . This occurs because in the baseline sampling all frequency components are acquired, but in the subtraction only those with f<<1/T are really effective for reducing the 1/f noise. At higher frequencies
  - components with $f = (2n+1) / 2T$ (n integer) have power enhanced x 4
  - components with $f = n\, 1/T$ (n integer) are canceled, power is zero
  - at the intermediate frequencies the power varies between zero and x 4 (see diagrams)

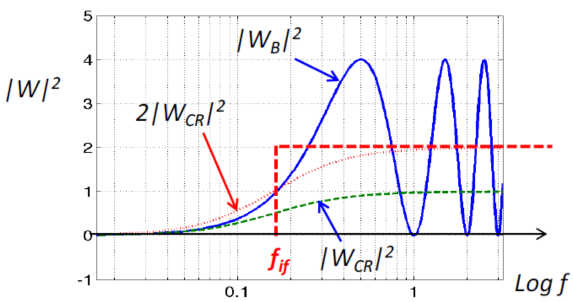
Summary

for convenience, the diagrams reported in slides are here repeated



LIN –LIN DIAGRAM
white noise power ∝ area of $|W|^2$

NB: examples with equal cutoff T=RC plotted for T=1



LIN –LOG DIAGRAM
1/f noise power ∝ area of $|W|^2$

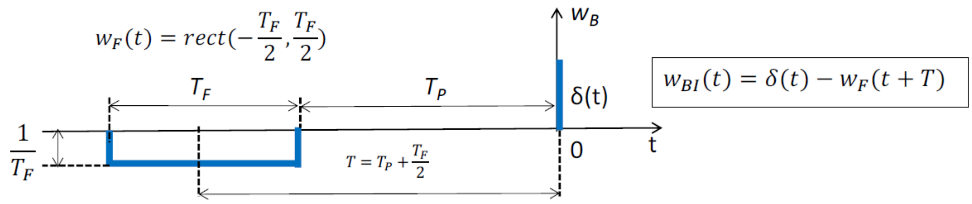
CDS WITH FILTERED BASELINE – CDS-FB

Do we really need a delta to acquire the baseline? The baseline is the offset, which ideally has a very LF. And we are acquiring something with a very LF with a delta, so acquiring a huge amount of noise to acquire something that is at LF. The ideal filter would be a LP filter → better to use a delta to acquire the signal and a rect to acquire the baseline. But the rect is GI, so as soon we increase the width of it in time domain, in frequency domain we are decreasing the BW and acquiring less noise.

The drawback is, from a practical point of view is that we need some time to acquire less noise (if we increase the width of the rect to reduce the noise).

So instead of using a delta for the baseline and a delta for the signal.

- **Baseline sampling is intended to acquire the contributions of the low-frequency** components that we want subtract from the measurement.
- However, **instant sampling acquires all frequency components** at low and high frequency; by subtracting them all, we double the noise passed above the CDS cutoff.
- **Remedy: modify baseline sampling for acquiring only the low-frequency components**; that is, sample with a low-pass weighting function $w_F(t)$ with band-limit f_{Fn} , which includes only the frequencies to be subtracted.
  Example: noise with upper bandlimit f_S and baseline acquired by a Gated Integrator with narrower filtering band $f_{fn} << f_S$ (recall $f_{f\,n} = 1/2T_F$ with gate duration T_F)



NB: we still consider cases with **long interval $T_P >> T_F$** from zero-setting to measurement

Now we have to write the equations.

$$W_{BI}(\omega) = F[w_{BI}(t)] = F[\delta(t) - w_F(t+T)] = 1 - e^{i\omega T} W_F(\omega)$$

since $W_F(\omega) = \text{sinc}\left(\frac{\omega\, T_F}{2}\right)$ is real at any ω , we have

$$W_{BI}(\omega) = 1 - W_F(\omega)\cos\omega T - iW_F(\omega)\sin\omega T$$
$$|W_{BI}(\omega)|^2 = 1 + W_F^2(\omega) - 2W_F(\omega)\ \cos\omega T$$

- At low frequency ($f << 1/T$) it is $W_F(f) \approx 1$ and W_{BI} has a high-pass cutoff equivalent to a CR differentiator with $RC=T$

$$\boxed{|W_B(\omega)|^2 \approx \omega^2 T^2 = \omega^2\left(T_P + \frac{T_F}{2}\right)^2}$$

$$\boxed{f_{if} \approx \frac{1}{2\pi T} = \frac{1}{2\pi\left(T_P + \frac{T_F}{2}\right)}}$$ cutoff frequency

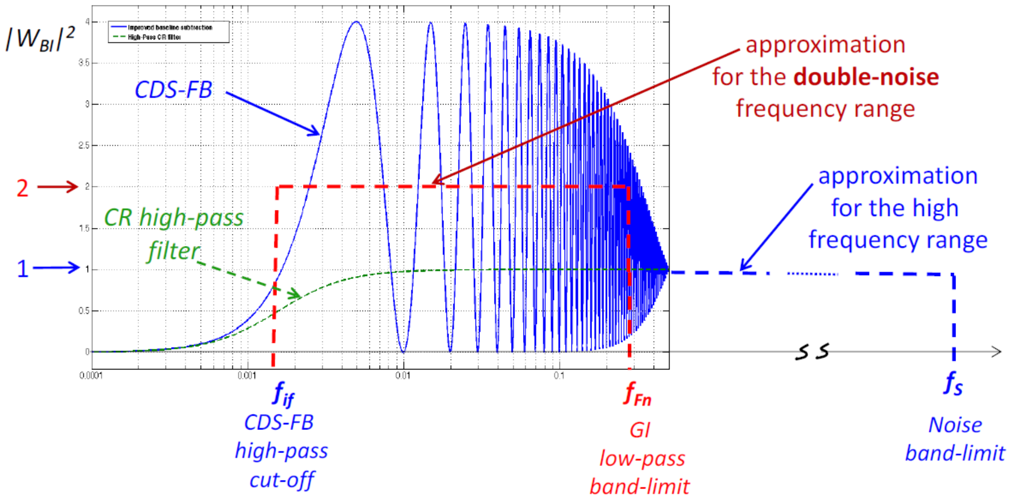
- At high frequency above the GI low-pass cutoff ($f >> f_n = 1/2T_F$) it is $|W_F(f)| \approx 0$ so that $|W_{BI}(f)|^2 \approx 1$
- In the intermediate range ($1/T << f << 1/2T_F$) it is roughly $W_F(f) \approx 1$ so that roughly it is $|W_{BI}(f)|^2 \approx 2(1 - \cos 2\pi f T)$. In this range the average value is about $|W_{BI}(f)|^2 \approx 2$, hence we can denote it as **double-noise range**

The w_f is the delta minus the w_f of a rect shifted. So the Fourier of the w_f is the Fourier of the delta minus the Fourier of the rect. The former is 1, the second is the one of the rect times the phase shift (we are shifting it in the time domain.

Since we made a change, we need to check the data, and our data is to find a cutoff at LF. We need to check if we have a cutoff at LF, because we removed the doubling of the noise but maybe also the LF cutoff.

The w_f when omega is really small (f << 1/T), the w_f of the rect is 1, so we have a high cutoff as in the normal CDS. If f > 1/T, the final w_f is 1.
So we here have a cutoff at very low frequencies, but in the CDS is a sinusoidal oscillation between 0 and 4 till to infinite, while here the w_f goes as the CR at HF.

In the middle we still have a doubling of the noise.



Example of CDS-FB with $T_P = 101$ and $T_F = 2$

for comparison, a CR filter with equal cutoff $RC = T = T_P + T_F$ is reported

At very LF we have the same behavoiur of the CDS that was the same behaviour of the CR. At very HF, it goes to 1, that is exactly the value of the CR. In the middle we have the sinusoidal with a factor 2.
We are focused on LF, but we should also consider the fs, HF cutoff, that is e.g. the frequency of the amplifier or of the filter to filter the signal, but now it's far at HF.
We have to extend the behaviour of the CDS-FB till fs (blue dashed). So I notice HP filter, doubling, one. So the blue curve is 1 after a certain point, in the CDS it has a factor 2 up to fs.

So we have a doubling of the noise in the red rect, that in the normal CDS is large as fs. We need to define some frequencies; the first one is the lower cutoff, and f_if is the frequency of the intrinsic HP filter. Then the second frequency is fs, which is the higher limit for the noise. Then f_Fn is the BW of the filter we introduced for the baseline (the rect for the baseline), so f_Fn is the BW of the GI.

I should make the integral of a factor 2 from f_if to f_Fn and of a factor 1 from f_Fn to fs. But instead of doing this, I do the integral from f_if to fs of the factor 1 and then in the previous interval I do the integral of a factor 1. This is for the computation of the output noise power.

Output nose power

$$\overline{n^2} = \int_0^{fs} S(f)|W_{BI}(\omega)|^2\, df = \int_0^{fs} S(f)[1 + W_F^2 - 2W_F\ \cos 2\pi f T]\, df$$

By approximating W_{BI} as outlined, the noise power can be approximately evaluated

1/f noise $\overline{n^2_{f,\,BI}} \approx S_B f_C \ln\left(\frac{f_S}{f_{if}}\right) + S_B f_C \ln\left(\frac{f_{Fn}}{f_{if}}\right)$

white noise $\overline{n^2_{B,BI}} \approx S_B(f_S - f_i) + S_B(f_{Fn} - f_i) \approx S_B f_S + S_B f_{Fn}$

In **CDS-FB the noise-doubling effect is strongly reduced** with respect to the simple CDS: it occurs only in the range from the low-frequency cutoff to the GI filtering band-limit.

In cases **where the GI band-limit is much smaller than the noise band-limit** ($f_S >> f_{Fn}$) the effect of noise doubling is practically negligible

$$\overline{n^2_{f,\,BI}} \approx S_B f_C \ln\left(\frac{f_S}{f_{if}}\right) \qquad \overline{n^2_{B,BI}} \approx S_B f_S$$

If we had two deltas, in CDS the f_Fn would be infinite, but if so we are limited by fs, so we return to the normal CDS, both for the 1/f and WN we get again the factor 2.

What is the real value of fs?
There is only one limitation, that fs has to be much larger than f_if, otherwise we cannot use the formula. So fs is any cutoff at HF, so we can imagine it as the cutoff of the filter to filter the signal, e.g. a GI.
The doubling is avoided when fs >> f_Fn, which means that we can neglect the red term, i.e. in the time domain I'm integrating for a large time, using a large rect so that f_Fn small. In this way we get the 1/f and WN without doubling.
This is why instruments take seconds to make the zero baseline.

This result is just the starting point, because we still have the 1/f noise, ok it is not infinite with the LF cutoff, but not leading to an optimal SNR. The problem is that f_i si dominated by the distance between the baseline setting and the measurement, and it can be very large, so I'm integrating a lot of 1/f noise.

To avoid this problem we could reduce Sb, but this cannot be done. Again, fc cannot be changed as soon as the fix the preamplifier. As for fs, it is the higher cutoff of the filter we will use, and with the optimum filter theory, the filter has to be the same shape of the signal, so the fs is the cutoff of the signal, so it is a data and we cannot change it. The only parameter we can change is f_i, but it has the problem that is very large.

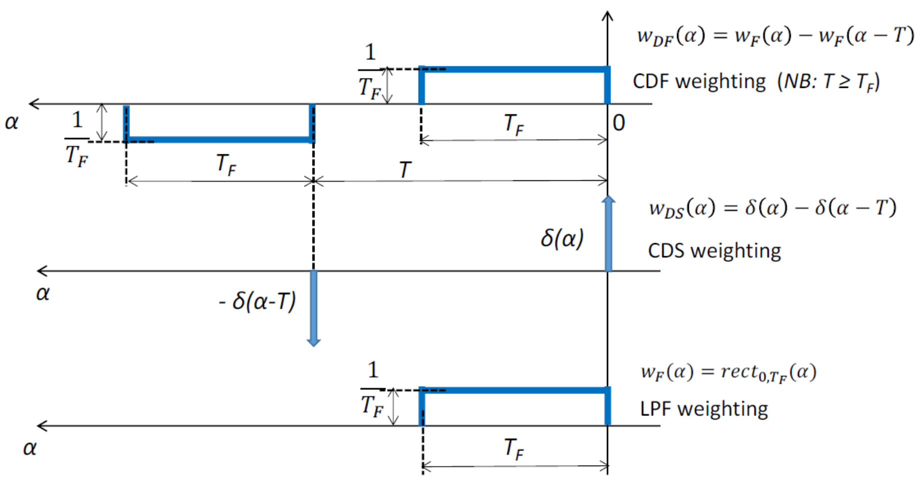
CORRELATED DOUBLE FILTERING - CDF

The WN and 1/f are described by a formula that has a limit at LF due to CDS and a limit at HF due to the filter that we are using. So to set the higher limitation we can use any type of filter instead of using a delta to acquire the signal. So we can study a particular case to make the zero as soon as we can.

- In various cases of pulse-amplitude measurements, **filtering by gated integrator (GI) is quite efficient** for the white noise component, but not for the 1/f component.
- An improvement is obtained by **subtracting from the GI acquisition of the pulse another GI acquistion over an equal interval** before the pulse (or after it, anyway outside the pulse)
- This approach has the **same conceptual foundation as CDS, but has the two samples filtered by the GI**: it is therefore called «Correlated Double **Filtering**» CDF
- The approach can be extended to cases where a constant-parameter low-pass filter LPF is employed for filtering the white noise component and a 1/f component is also present
- In such cases, the measure can be obtained as a difference of two samples of the LPF output: a sample taken at the pulse peak and a sample taken before the pulse (or after it, anyhow outside the pulse)

If we have the sync and know where the signal is, we can think of making the zero and making the measurement of the signal then. So we would like to use a rect also for the signal, not only for the baseline. If we use this approximation, we get exactly the same formula as before. In this case fs = f_Fn because the rects have the same width, so we are doubling the noise because we are using the same signal for the filter and for the noise.

The difference is in the fact that the BW I have to use in the rect for the baseline should be much smaller than the one I use for the rect of the signal. In this way I can avoid the doubling of the noise. If we use the same BW of course we are doubling the noise. We study a particular case where T, the distance between the two rect, that is really small. This is a case not included in the previous calculations, because one of the hypothesis of the previous calculations was that fs is much larger than fi.



CDF weighting = convolution of CDS weighting with LPF weighting

$$w_{DF}(\alpha) = w_{DS}(\alpha) * w_F(\alpha)$$

The w_f in the time domain is the convolution of the w_f of the CDS times the w_f of the LP filter, and in this case the LP filter is a gated integrator. If we put all together the convolution in the time domain is the product in the frequency domain, but we are interested in the squared product of the absolute values.

Since in time domain $\quad w_{DF}(\alpha) = w_{DS}(\alpha) * w_F(\alpha)$

in frequency domain it is $\quad W_{DF}(\omega) = W_{DS}(\omega) \cdot W_F(\omega)$

for noise computation $\quad |W_{DF}|^2 = |W_{DS}|^2 \cdot |W_F|^2$

and since $\quad |W_{DS}|^2 = 2(1-\cos\omega T) = 4\sin^2(\omega T/2)$

we have

$$\boxed{|W_{DF}|^2 = 2(1-\cos\omega T)\cdot|W_F|^2 = 4\sin^2\left(\frac{\omega T}{2}\right)\cdot|W_F|^2}$$

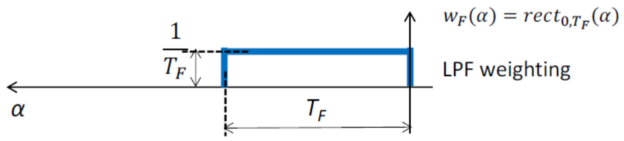
The main features of CDS reflect the fact that it is a combination of CDS and LPF :

1. The **LPF cuts the noise at high frequencies** with its LPF band-limit f_F
2. The **CDS cuts the noise at low frequencies** with its HPF band-limit $f_{iD} \approx 1/2\pi T$
3. The **CDS enhances the noise in the passband between the band-limits** (with enhancement factor roughly 2)

For sure we will have a lower cutoff due to CDS and a higher cutoff due to LP filter action, but the cutoff of fs and of the LP filter that we used with the baseline is the same, because we are using the same rect for the signal and the baseline.

The problem is that not only we collapse the two cutoffs of fs and fn, but we also collapse fs to a value that is really similar to the value of the lower cutoff. At this point we have no more the LF cutoff and the HF cutoff well spaced so we can use the normal formula.

If we put all together (image below), the Fourier of the rect is the sinc and we get the function in the box of the next image.



$$w_F(\alpha) = rect_{0,T_F}(\alpha) = rect_{-\frac{T_F}{2},\frac{T_F}{2}}\left(\alpha - \frac{T_F}{2}\right) \rightleftarrows W_F(\omega) = Sinc\left(\frac{\omega T_F}{2}\right)e^{-j\omega\frac{T_F}{2}}$$

but the module does not depend on the phase factor (i.e. on the time shift)

$$|W_F(\omega)| = \left|Sinc\left(\frac{\omega T_F}{2}\right)\right| = \left|\frac{sin\left(\frac{\omega T_F}{2}\right)}{\frac{\omega T_F}{2}}\right|$$

Therefore

$$|W_{DF}|^2 = 2(1-\cos\omega T)\cdot|W_F|^2 = 2(1-\cos\omega T)\cdot Sinc^2\left(\frac{\omega T_F}{2}\right) = 4\sin^2\left(\frac{\omega T}{2}\right)\cdot Sinc^2\left(\frac{\omega T_F}{2}\right)$$

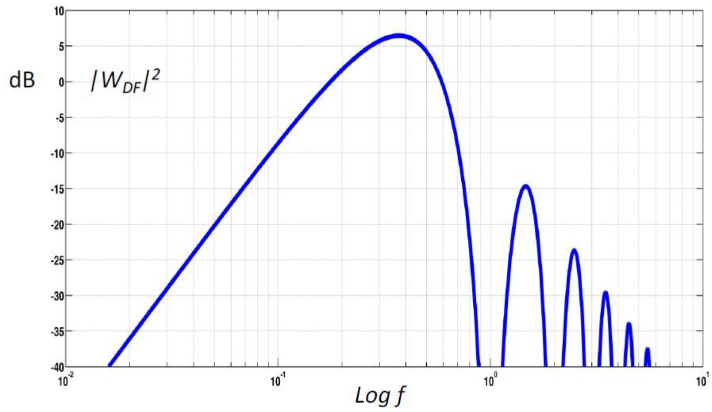
that is

$$\boxed{|W_{DF}|^2 = 2(1-\cos\omega T)\cdot\frac{sin^2\left(\frac{\omega T_F}{2}\right)}{\left(\frac{\omega T_F}{2}\right)^2} = 4\cdot\frac{sin^4\left(\frac{\omega T_F}{2}\right)}{\left(\frac{\omega T_F}{2}\right)^2}}$$

The shape has a cutoff at LF, like the HP filter, but instead of a part that doubles and another that has gain 1, it dies. The problem is that we don't have a distance between lower cutoff and higher cutoff and so we cannot use the previous formula.

So we can do two things; either we plot with MATLAB the w_f and we compute the noise (area of the function) or, since I have to use the filter just one time, I do an approximation, because the use of two rects very close one to the other (0 distance, at the end of on rect it starts the other one) is something we will use several time. And if we reduce the distance between the baseline and the signal we are reducing the T and we are increasing the cutoff for the 1/f. So the higher limit at LF for the 1/f is when signal and baseline are at zero distance (T = Tf, the distance is equal to the width of the rect).

Computed for the case of time shift $T = T_F$ integration time = 1



In this situation the shape is the one above. I notice that there is one big lobe and several small ones, whose area is probably negligible. So the ide is to approximate the main lobe with a rect.

MEASURING PULSE SIGNALS WITH 1/f NOISE

At this moment we didn't solve the issue of the 1/f, we understood that we can make the measurement with 1/f without doubling the noise and without having a SNR = 0.
The optimum filter theory doesn't work because the noise is not white. The problem is that it is impossible to create a whitening filter for the 1/f noise → we cannot use the optimum filter theory. In principle we cannot manage this situation.

But we can solve with the idea that the 1/f noise is logarithmic with the BW and connected to the ratio between HF and LF, while the white noise is connected with the difference of HF and LF.

Case: amplitude measurement of **pulse** signals with **1/f and wideband** noise.

The **classic approach** to optimum filtering (to find first a noise-whitening filter and then a matched filter) **is arduous in this case because 1/f noise**

- sets a remarkably difficult mathematical problem
- makes the whitening filter difficult to design, not implementable with lumped circuit components, but with distributed parameters (distributed RC delay lines, etc.)

However, by noting that

a) for **1/f noise** the filtered power

- mainly depends on the span of the band-pass measured by the **bandlimit ratio**, hence it is **markedly sensitive to the lower bandlimit level**
- **weakly** depends on the **shape** of the filter weighting function

b) for **wideband noise** the S/N

- depends on the span of the band-pass measured by the **bandlimit difference**, hence it is **weakly sensitive to the lower bandlimit level**
- markedly depends on the shape of the weighting function

an alternative approach leading to quasi-optimum filtering can be devised

The first step is to completely remove the 1/f. Then we compute the whitening filter (e.g. if we have 1/f^2) and the optimum filter and so on. If we are lucky, this is enough.

FIRST STEP:

- Design a **main filter** for signal and wideband noise only (that is, considering non-existent the 1/f noise) and then
- Take then into account the 1/f component and evaluate the **additional noise power** that 1/f noise brings to the main filter output.

In the (lucky) cases where this 1/f noise power is smaller than the wide-band noise (or at least comparable), the main filter may be considered sufficient without further filtering.

Otherwise, if the addition due to 1/f noise is excessive, proceed to the

SECOND STEP :

- design an **additional filter** for limiting the 1/f noise power without worsening excessively the filtering of the wideband noise.

It is obviously a **high-pass filter**, which must combine the goal of
a) reducing efficiently the 1/f noise power

with the further requirements of
b) **limiting** to tolerable level the **increase of the filtered wide-band noise**
c) **limiting** to tolerable level the **reduction of the output signal amplitude**

As a second step we try to compute the effect of the 1/f on the filter we chose without considering the 1/f noise. If it is negligible compared to WN we are finished.

If not, we try to add an additional filter to try to reduce the effect of 1/f. Obviously, the additional filter will be a HP filter, because the LP filter is for the signal, but how can I choose it?
First of all, the additional HP filter has to efficiently reduce the 1/f noise and we don't want to damage the signal, because if we apply this signal after the matched filter we are applying it to both the signal and the noise.

The problem is that we apply the filter to both the signal and the noise, so we want to cut the 1/f noise but also have a small impact on the signal. Furthermore, adding a filter to remove the 1/f could have an effect on the signal (CDS theory) cutting a part of it, but also eventually enhancing the white noise.

FIRST STEP

We are neglecting the 1/f noise, so we can find the optimum filter since we have just WN, so the optimum filter is the matched filter low pass filter. As an approximation, we suppose that the LP filter in the frequency domain can be approximated with a rect and we are interested in the value in 0 in the time domain of the signal, because it is a very low frequency signal.
The value in 0 of the signal is the integral of the Fourier transform in the frequency domain, which is the amplitude times the BW of the filter, because we approximated the matched filter with a rect in the frequency domain.

The issue is better clarified by considering as FIRST STEP the **optimum filter for signal and wide-band noise (or its approximation)** composed by

- Noise-whitening filter, with output white noise S_B and pulse signal.
  Let f_S be the upper band-limit and A the center-band amplitude of the pulse transform.
- Matched filter, which has weighting function matched to the pulse signal from the whitening filter and is therefore a low-pass filter with upper bandlimit f_S.
  The output has a signal with amplitude roughly $V_S \approx A\, f_S$ and band-limited white noise with band-limit f_S and power

$$\overline{n_B^2} \approx S_B f_S$$

For focusing the ideas, let's consider a well known specific case: filtering of pulse-signals from a high impedance sensor with an approximately optimum filter, i.e. with matched filter approximated by a constant-parameter RC integrator.
In this case, the output noise corresponding to the input wide-band noise is a white noise spectrum with band-limit set by a pole with time constant $RC=T_{nc}$

Now we have the WN as Sb*fs, and the BW limitation for the noise is the same for the signal because we are using the matched filter.
Then we have to introduce the 1/f noise to understand if we are lucky or not. Since we are using a matched filter with the same BW of the signal, it is not strange that both signal and WN have the same BW.

SECOND STEP

We are introducing the 1/f noise to see if its effect is negligible compared to the WN.

Let's now take into account also a 1/f noise source, which brings at the whitening filter output a significant 1/f spectral density $S_B f_C/f$.

At high frequency, the 1/f component is limited by the upper bandlimit f_S of the matched filter.

At low frequency, the 1/f component can be limited by a lower band-limit f_i set by an additional constant-parameter filter. With $f_i << f_S$ the output power of the *1/f* noise can be evaluated as

$$\overline{n_{fn}^2} \approx S_B f_C \ \ln\left(\frac{f_S}{f_i}\right)$$

However, the constant-parameter high-pass filter operates also on the signal: it attenuates the low frequency components and thus causes a loss in pulse amplitude, hence a loss in S/N. The reduced amplitude is roughly evaluated as

$$V_S \approx A(f_S - f_i) = Af_S\left(1 - \frac{f_i}{f_S}\right)$$

For limiting the signal loss, f_i/f_S must be limited; e.g. for keeping loss < 5% it must be

$$\frac{f_i}{f_S} \leq 0{,}05 \qquad \text{that is} \qquad \ln\left(\frac{f_S}{f_i}\right) \geq 3$$

The effect of 1/f noise is Sb*fc*ln(fs/fi), if fs >> f_Fn. Sb, fc and fs are data, fi has to be chosen.
So in theory I have just to choose the correct value for fi. The problem is that if fs = fi we are loosing part of (all) the signal, so the signal is 0. The signal is A(fs-fi) because we are cutting the low frequencies, that is the original signal A*fs minus a factor.
If we want to loose e.g. only 5% of the signal, f1/fs < 0.05, so ln(fs/fi) > 3.

The goal is to have 1/f noise much smaller than the WN.

For reducing the 1/f noise to the white noise level or lower

We need that

$$S_B f_C \ \ln\left(\frac{f_S}{f_i}\right) \leq S_B f_S$$

$$f_C \ \leq \frac{f_S}{\ln\left(\frac{f_S}{f_i}\right)}$$

and since for keeping the signal loss <5% it must be $\ln\left(\frac{f_S}{f_i}\right) \geq 3$

we need to have

$$f_C < \frac{f_S}{3}$$

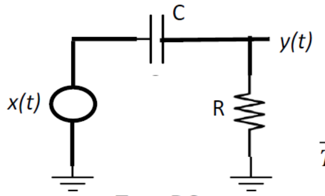
This means that the goal can be achieved only if the 1/f noise component is low or moderate. Note that **f_C and f_S are data** of the problem, they cannot be changed. In cases where f_C exceeds the above limit, a constant-parameter high-pass filter is NOT a suitable solution for reducing the 1/f noise power.

CONCLUSION: constant-parameter high-pass filters can be useful as additional filter for limiting the 1/f noise, but just in cases with moderate 1/f noise intensity, because of their detrimental effect on the signal pulse amplitude.

In order to have a 1/f noise negligible compared to the WN we need a fc that is fc < fs/3. At this point we don't have any other parameter to choose if the text gives us this condition satisfied. If not, the 1/f is dominant.

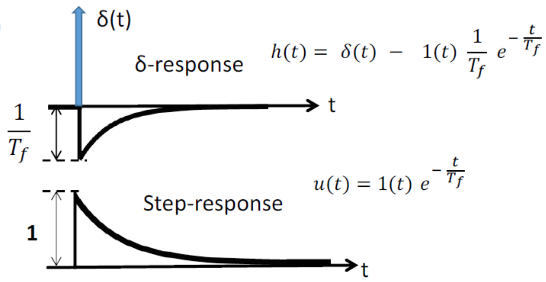
If so, we need to add another filter.

BASIC CONSTANT PARAMETER HP FILTER (CR DIFFERENTIATOR)



$$T_f = RC$$

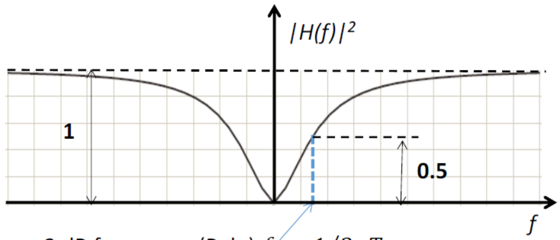
$$f_p = \frac{1}{2\pi T_f}$$



Transfer function

$$H(f) = \frac{j\,2\pi f T_f}{1 + j\,2\pi f T_f}$$

$$|H(f)|^2 = \frac{(2\pi f T_f)^2}{1 + (2\pi f T_f)^2}$$



3 dB frequency (Pole) $f_p = 1/2\pi T_f$

Firstly, let's study the basic CPF that the HP filter is. We have one zero in the origin and one pole. The bottom right plot is the w_f in the lin-lin plot. **The delta response is the derivative of the step response**.

We can write the HP filter like the all pass filter minus the law pass filter, always. This can be demonstrated with the formulas in time frequency domain or time domain. The all pass is 1.

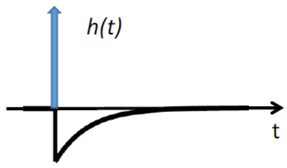
The intuitive view

«High-Pass Filter = All-Pass - Low-Pass Filter»

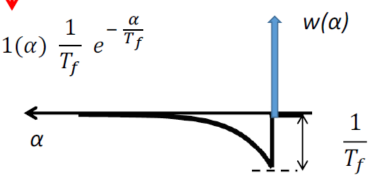
is confirmed by

Transfer function $\quad H(f) = 1 - \dfrac{1}{1 + j\,2\pi f T_f} = \dfrac{j\,2\pi f T_f}{1 + j\,2\pi f T_f}$

δ-response $\quad h(t) = \delta(t) - 1(t)\,\dfrac{1}{T_f}\,e^{-\frac{t}{T_f}}$



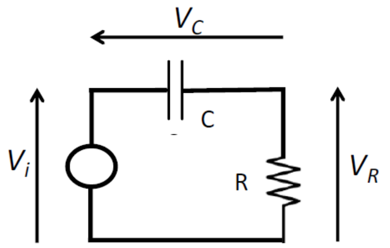
Weighting function $\quad w(\alpha) = \delta(\alpha) - 1(\alpha)\,\dfrac{1}{T_f}\,e^{-\frac{\alpha}{T_f}}$



From the Kirchhoff standpoint:

The circuit mesh structure itself confirms that

«High-Pass Filter = All-Pass - Low-Pass Filter»



V_i = input voltage
V_C = low-pass filtered V_i
V_R = high-pass filtered V_i

Kirchoff's mesh voltage law

$$V_i = V_C + V_R$$

Therefore $V_R = V_i - V_C$

that is

High-pass filtered V_R = resistor voltage =
= input voltage V_i – capacitor voltage =
= input voltage V_i – Low-pass filtered V_i

Seeing the HP filter as the difference of the all pass and LP filter saves time in the computations. We want to compute the ENBW for the HP filter.
If I have WN and HP filter and I want to compute the power of the noise as if it was a rect, the ENBW is pi/2 * fp, because if all pass is constant and the LP is a rect at a frequency fp, then the HP is the equivalent rect.

High-pass band-limit for White noise

Premise: with only a high-pass CR filter the white noise power $\overline{n_B^2}$ is divergent, therefore we consider here also a low-pass filter with band-limit $f_s >> 1/RC$.

The high-pass band-limit f_i of the CR filter with weighting function *W(f)* is defined by

$$\overline{n_B^2} = S_B \int_0^{f_S} |W(f)|^2 df = S_B \overset{x}{\int_0^{f_S}} \frac{\left(\frac{f}{f_p}\right)^2}{1+\left(\frac{f}{f_p}\right)^2} df = S_B(f_S - f_i)$$

The computation of the integral can be avoided by recalling that

CR high pass filter = all-pass – RC low-pass filter

and therefore

high-pass band-limit f_i of the CR filter = low-pass band-limit f_h of the RC filter

$$f_{iCR} = f_{h\ RC} = \frac{1}{4RC}$$

We wat to rewrite the integral x as Sb(fs – fi) and compute fi to write the noise as a rect. For the WN the limits for HP and LP are both 1/(4RC), and we want to find them writing the noise of the 1/f as a rect also for the 1/f noise.

High-pass band-limit for 1/f noise

Premise: with only a high-pass CR filter the 1/f noise power $\overline{n_f^2}$ is divergent, therefore we consider here also a low-pass filter with a high band-limit f_s >> 1/RC.

The high-pass band-limit f_{if} of the CR filter is defined by

$$\overline{n_f^2} = S_B f_c \int_0^{fs} \frac{\left(\frac{f}{f_p}\right)^2}{1+\left(\frac{f}{f_p}\right)^2} \frac{df}{f} = S_B f_c \int_{f_{if}}^{fs} \frac{df}{f} = S_B f_c \ln\left(\frac{f_s}{f_{if}}\right)$$

In this case the first integral is fairly easily computed and shows that

$$f_{if} = \frac{f_p}{\sqrt{1+\left(\frac{f_p}{f_s}\right)^2}}$$

that is, for $f_s >> f_p$

$$f_{if} \approx f_p = \frac{1}{2\pi RC}$$

The idea is that if we use a HP filter, let's compute the 1/f noise n_f^2 and we want to approximate the computation as a rect, as an integral from f_if and fs. This equation is true only if we have a sharp cutoff (rect).

I want the f_if value that allows me to write it as a sharp rect with sharp cutoffs. So with f_if I have to choose the exact frequency of the pole if fs >> fp.

Extended calculations

$$\overline{n_f^2} = S_B f_c \int_0^{fs} \frac{\left(\frac{f}{f_p}\right)^2}{1+\left(\frac{f}{f_p}\right)^2} \frac{df}{f} = S_B f_c \frac{1}{2} \int \frac{g'(f)}{g(f)} df$$

Considering $\quad g(f) = 1 + \left(\frac{f}{f_p}\right)^2 \quad$ and $\quad g'(f) = 2\frac{f}{f_p^2}$

We can solve the intregral by substitution obtaing:

$$\overline{n_f^2} = S_B f_c \frac{1}{2} \ln\left(1+\left(\frac{f_s}{f_p}\right)^2\right)$$

And then make it equal to the final form:

$$\overline{n_f^2} = S_B f_c \frac{1}{2} \ln\left(1+\left(\frac{f_s}{f_p}\right)^2\right) = S_B f_c \ln\sqrt{\left(1+\left(\frac{f_s}{f_p}\right)^2\right)} = S_B f_c \ln\left(\frac{f_s}{f_{if}}\right)$$

$$f_{if} = \frac{f_s}{\sqrt{1+\left(\frac{f_s}{f_p}\right)^2}} = \frac{\frac{f_p}{f_s}}{\frac{f_p}{f_s}} \frac{f_s}{\sqrt{1+\left(\frac{f_s}{f_p}\right)^2}} = \frac{f_p}{\sqrt{1+\left(\frac{f_p}{f_s}\right)^2}}$$

The important thing is that when computing the noise power for the 1/f, f_i I have to choose the frequency of the pole for the HP filter case.

So if I have to choose the frequency of the pole, so 1/(2*pi*RC) for the HP filter in the formula of the ln compute the 1/f, if I have the CDS, which is the value to put in the formula?
1/(2*pi*T), because at lower cutoff the CDS is equal to a CR with T = RC.

SUMMARY

- **The upper frequency limit f_S :**
  - is necessary for limiting the white noise power
  - is useful also for limiting the 1/f noise power
  - the level of f_S is dictated by the pulse signal to be measured

- **The lower frequency limit f_i :**
  - is necessary for limiting the *1/f* noise power,
  - the selected level of f_i is conditioned by the pulse signal, it cannot be arbitrary
  - however, the reduction of *1/f* noise is significant even with fairly low f_i , that is, with f_S/f_i values that are high, but anyway finite.

The important thing is that the level of fs is dictated by the pulse signal to be measured; we cannot choose fs because fs is connected to the signal and we want to collected the signal and limit the amount of noise. As for fi it is necessary only to limit 1/f, normally it has no effect for WN.
If we area applying a filter at the end of the chain that acts both on the signal and the noise, the higher the frequency of the HP filter, the lower the 1/f but also the lower is the value of the signal.

However, 1/f is cut significantly even with low fi. Changing fs has not a really big effect, but changing fi has, because in principle 1/f starts from 0, so if instead of 0 we are taking a very close value for fi we are limiting the 1/f a lot. From 1.001 to 1.1 Hz we change a lot the 1/f noise, even if we are still at 1 Hz. For the WN this is not an issue.

But we cannot simply use a HP filter to cut the 1/f, only if we are lucky that frequency corner is small and we don't want to loose a lot of signal. But there is another problem with the HP filter.

If the signal is a pulse, sometimes also if fs is lower than fs/3, we have problems because in the real world we never have just one pulse, but a sequence of them. Normally we are happy to have a sequence of pulses, because if they have the same amplitude (or similar), we can filter different pulses to increase the SNR. So if we can repeat the measurement we are happy.
The point is that if we have a sequence of pulse with a CP filter (CR filter), something happens.

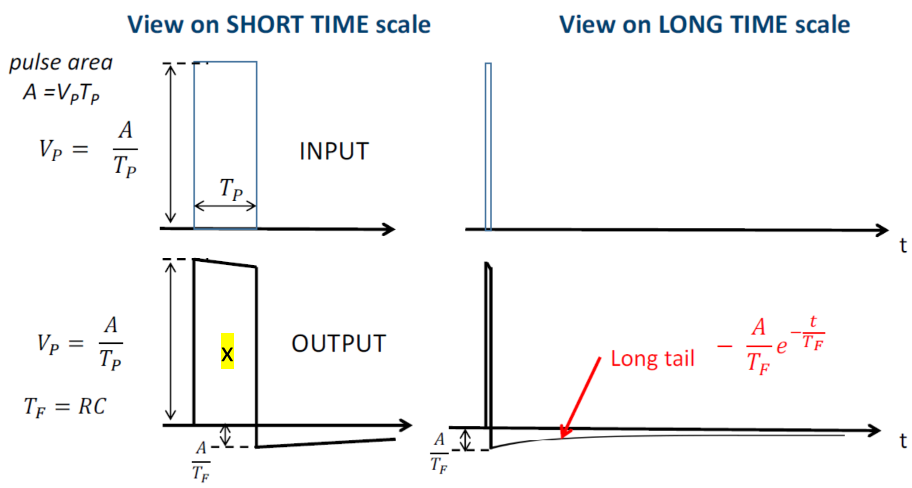
CR FILTER AND PULSE SEQUENCE

So we have our pulse, and for the problem of the 1/f we need to use a HP filter. If we pass a rect in the HP filter, if RC is much larger than Tp we get x. Normally we neglect the tail if the tau is very large, and the rect at the output is the same of the rect at the input, so we neglect the tail.

In general, in the time domain the response of a filter is the convolution between the delta response of the filter and the signal, that in the frequency domain translates into the product of the Fourier of the filter times the Fourier of the signal. As for the Fourier of the HP filter, it is the HP filter; the value in 0 of the Fourier transform of the CR HP filter is 0.
When we multiply the Fourier transform of the filter and the one of the signal, if the Fourier of the filter has 0 in the origin, the result has to have a zero in the frequency domain. But zero in the frequency domain is the area in the time domain.

So every time we apply a signal to a HP filter, the output has to have zero area, because in the frequency domain we have a zero. So the area of the tail is exactly equal to the area of the rect. If we increase the tau, the difference between the beginning and the end of the rect is really small, but at the same time the length of the tail is very long (right plot).

Let's look in detail the effect of a high-pass filter (RC = T_F) on a pulse signal

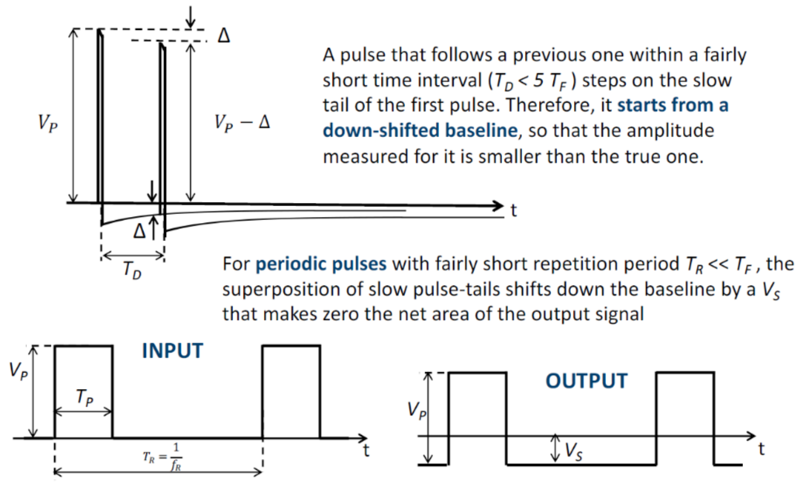


NB: DC transfer of CR is zero → net area of the output signal is zero

One of the problems of the CR is that there is a tail when we have a very long tau, but at the output of the filter the area must be 0 because it is a HP filter. The ling tail compensate exactly the area of the positive part of the previous image, and it lasts for a long time. The problem is not if we use a single pulse, but if we have a sequence of pulses

CR FILTER AND SEQUENCE OF PULSES

The first pulse creates the tail, and the second pulse starts from the tail of the first one, not from 0, and has a tail itself that sums to the previous one.



Repetition-rate-dependent baseline-shift $V_S = V_P \frac{T_P}{T_R} = A\, f_R$

If the input is a square waveform the sum of the tail is so important that we have a shift of the waveform, and this is correct because the positive part has to be equal to the positive one. For a periodic signal is not so bad to have a shift because the area has to be 0 at the output, because I can compute the shift, it is deterministic.

The high-pass filtering (differentiator action) of the CR filter has **MIXED effects.**

- The effect **on noise is ADVANTAGEOUS:** by cutting off the the low frequencies it markedly decreases the 1/f noise power (and mildly reduces the white noise power)
- The effect **on the signal is DISADVANTAGEOUS**:
  - it **decreases the signal amplitude** by cutting off the low frequencies of the signal , hence f_i must be kept low ($f_i << f_S$ of the pulse) in order to limit the signal loss. However, this limits also the reduction of 1/f noise
  - it **generates slow tails after the pulses**, which shift down the baseline and thus cause an error in the measured amplitude of a following pulse
  - With a **periodic** sequence of equal pulses, all pulses find the **same baseline shift**. The amplitude error is constant, sistematically dependent on the repetition rate.
  - With **random-repetition** pulses (e.g. pulses from ionizing radiation detectors) the pulses occur randomly in time. Hence the random superposition of tails produces a **randomly fluctuating baseline shift**. The resulting amplitude error is random: in this case the effect is equivalent to that of an additional noise source.

CONCLUSION: a differentiator action is **desirable on noise**, but **NOT on the signal.**

WANTED: not a constant-parameter differentiator, but a true **Base-Line Restorer (BLR)**

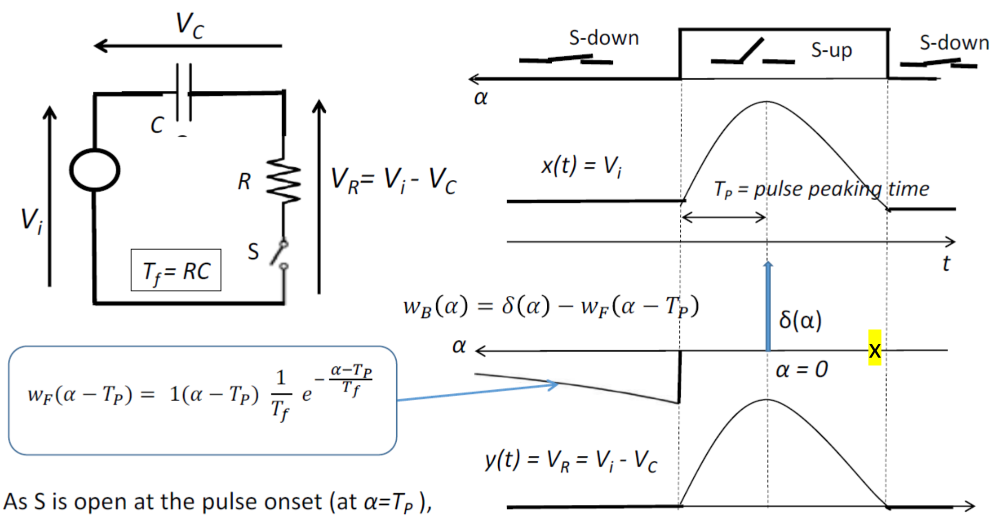
The problem is that if the signal is not periodic but stochastic, we don't know when the second pulse will come, so I don't know delta and the real amplitude I'm measuring, I'm making an error of a quantity delta.
Hence the CR is good because it is cheap, and compared to CDS we don't have doubling of the noise. But if we use a CR we act both on the noise and the signal, so we remove part of the signal. Moreover, it creates tails and the tails are tolerable with periodic signals but not with stochastic signals, with the latter we cannot make measurements.

So the CR works perfectly for the noise but not for the signal, so the idea is to introduce a baseline restorer, a NCP filter that is a CR only for the noise.

BASELINE RESTORER

We start from a CR and we add a switch. Every time we add a switch we need to define where to put it and how it works. Since we would like to have a CR only on the noise, we have to close the switch when there is no signal, so that we have a CR.

High-pass filtering action **on the noise** and **NOT on the signal: switched-parameter** CR filter with CR → ∞ when signal is present, finite CR = T_F when no pulse is present

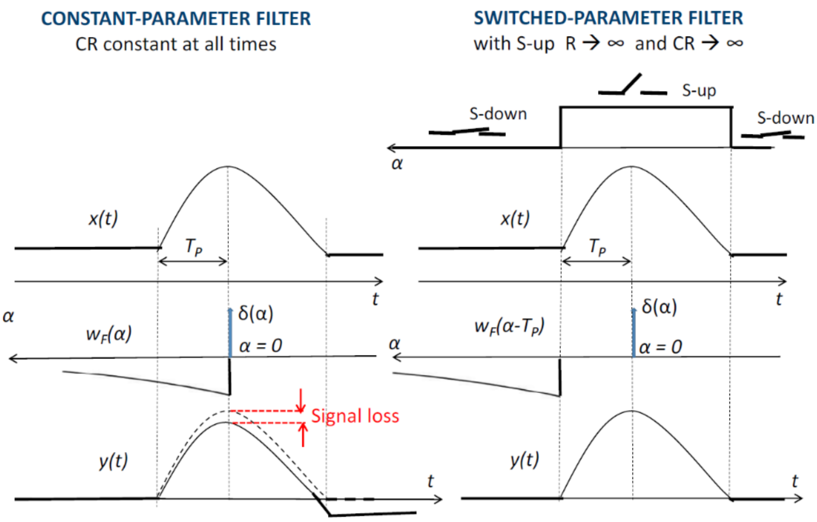


As S is open at the pulse onset (at $\alpha = T_P$),
charging of C stops and voltage V_C stays constant at the stored value

The weighting function of the CR with the switch is x. When the switch is open, if I apply a delta to create the w_f, the effect of the delta on t_m is 0, but there is one time where there is an effect, that is exactly at t_m, this is the reason why there is a delta in the w_f for t_m, that is a value I choose (typically in correspondence of the maximum).

The w_f in the blue box is a delta minus the w_f of a LP filter, which is exactly the idea of the HP filter (1 – LP). The difference is that we include the shift, because the switch gives us the possibility to shift where to take the 1 and the LP filter.

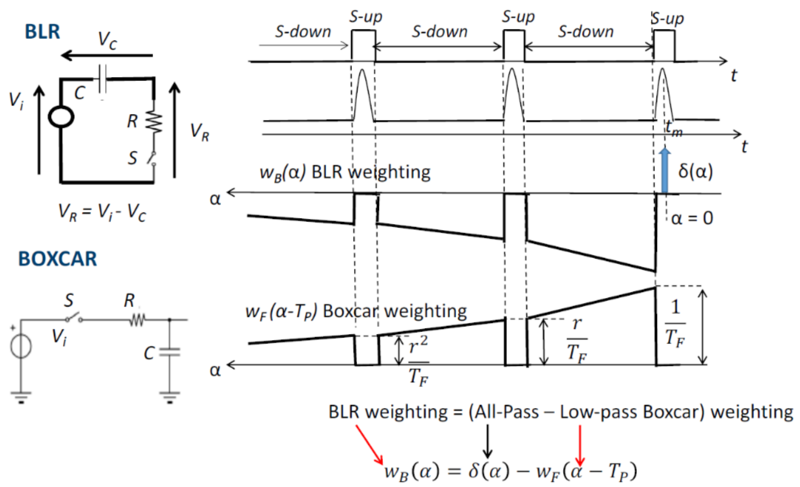
CR FILTER AND BLR COMPARISON



So the CR works also on the signal, but the BLR has the same w_f of the CR but the negative LP filter and the delta are shifted, so the signal is exactly the same.

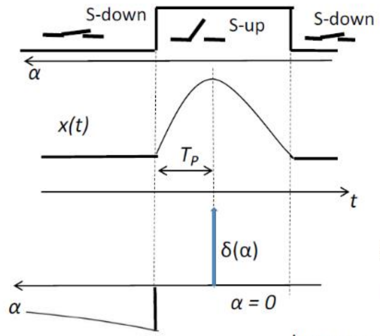
Now we increase the complexity of the filter because the real problems of the CR were that it acted on the signal, damaging it (remover with the BLR), but it had also a problem with periodic pulses. Can I solve this issue also with the BLR?

Instead of opening the switch one time, I open it every time I have the signal. If the tau of the LP filter finishes between two different pulses I'm applying the same filter multiple times. The good thing is that as soon as I open the switch when I have the signal I'm not necessarily forced to have a CR that finishes between two pulses, because I can have a CR also with a long tau because the tau 'doesn't touch' the signal.



Since the HP filter can be written as 1 – LP and 1 in the time domain is a delta minus a boxcar shifted, in the end we get the reverse of the boxcar integrator.

BLR weighting in frequency



BLR principle is alike filtered zero-setting, but with a **basic advantage:** much **shorter T_P** much **higher band-limit f_{if} (high-pass)**

(the BLR switch is electronically controlled, the interval T_P can be very short)

BLR weighting = All Pass – Low-pass

$$w_B(\alpha) = \delta(\alpha) - w_F(\alpha - T_P)$$

Low-pass weighting in frequency:

$$W_F(\omega) = F[w_F(\alpha)] = R_F(\omega) + i\, I_F(\omega)$$

BLR weighting in frequency:

$$W_B(\omega) = 1 - e^{j\omega T_P} W_F(\omega) = 1 - [\cos\omega T_P - j\sin\omega T_P]\cdot[R_F + jI_F] =$$
$$= [1 - R_F\cos\omega T_P - I_F\sin\omega T_P] - j[I_F\cos\omega T_P - R_F\sin\omega T_P]$$

Having more than one parameter allows us to choose the tau of the filter and where to close the switch. I want to get how to set the parameters.

So the w_f is a delta (1) minus the w_f of a LP filter. The problem is that we are using a generic w_f for the LP filter, which I don't know if it is real or a complex function. So I have to add also the complex part to take into consideration all the possible cases.
The steps are the same of the CDS, and we are splitting the real part and complex part, because we want the absolute value squared of the w_f. Let's carry on the calculations.

BLR weighting for noise:

$$|W_B(\omega)|^2$$
$$= [1 - R_F\cos\omega T_P - I_F\sin\omega T_P]^2 + [I_F\cos\omega T_P - R_F\sin\omega T_P]^2 =$$
$$= 1 + R_F^2 + I_F^2 - 2R_F\cos\omega T_P - 2I_F\sin\omega T_P =$$
$$= 1 + |W_F|^2 - 2R_F\cos\omega T_P - 2I_F\sin\omega T_P$$

Let's consider just cases where the interval between pulses is much longer than T_F so that

$$w_F(\alpha) = 1(\alpha)\,\frac{1}{T_f}\,e^{-\frac{\alpha}{T_f}} \quad \text{and} \quad W_F(\omega) = \frac{1}{1 + j\omega T_F}$$

and therefore

$$|W_B(\omega)|^2 = 1 + \frac{1}{1+\omega^2 T_F^2} - 2\frac{1}{1+\omega^2 T_F^2}\cos\omega T_P + 2\omega T_F\cdot\frac{1}{1+\omega^2 T_F^2}\sin\omega T_P$$

For the LP filter we use the standard RC filter, then we put this in the original one. The unknowns are Tf, Tp and Rn.
Since it is too complex, we can try to simplify the function and study only specific frequencies. We can go for instance to low frequencies, but since we have both Tf and Tp, what does lower mean?
Let's go for Tp.

In the low-frequency region $\omega \ll \dfrac{1}{T_P}$ with the approximations

$$\sin \omega T_P \approx \omega T_P \qquad \cos \omega T_P = 1 - \frac{\omega^2 {T_P}^2}{2}$$

we get

$$|W_B(\omega)|^2$$
$$\approx 1 + \frac{1}{1+\omega^2 T_F^2} - \frac{2}{1+\omega^2 T_F^2} + \frac{\omega^2 T_P^2}{1+\omega^2 T_F^2} + 2\frac{\omega^2 T_P T_F}{1+\omega^2 T_F^2} =$$
$$= \frac{\omega^2 (T_P + T_F)^2}{1+\omega^2 T_F^2} = \frac{\omega^2 T_F^2}{1+\omega^2 T_F^2}\left(1 + \frac{T_P}{T_F}\right)^2$$

and in the lower region $\omega \ll \dfrac{1}{T_F} \ll \dfrac{1}{T_P}$

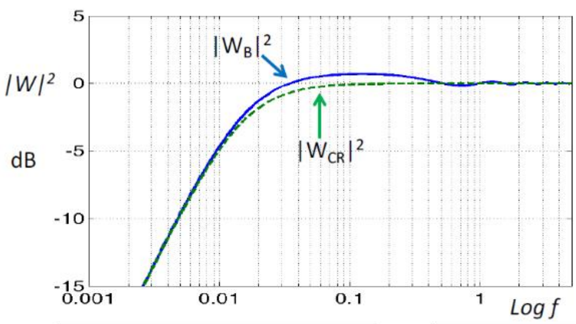
$$|W_B(\omega)|^2 \approx \omega^2 (T_P + T_F)^2$$

That is, the BLR has a cutoff equivalent to a CR high-pass with RC= T_P+T_F

For just w << 1/Tp, we still get something complicated, so let's go for w << 1/Tp and 1/Tf. At very low frequency, the BLR seems equal to a CR with T = Tp + Tf.
This is a good thing from the noise standpoint, because a cut at LF is something we like.

BLR vs CR HP FILTER – CUT OFF

Looking at the Bode plot at LF, the BLR is similar to a CR, so it works well. At HF, the curve is equal to the CR, which is good. However, we have no more a sinusoidal as in the CDS but an overshooting, oscillations and then to 1.

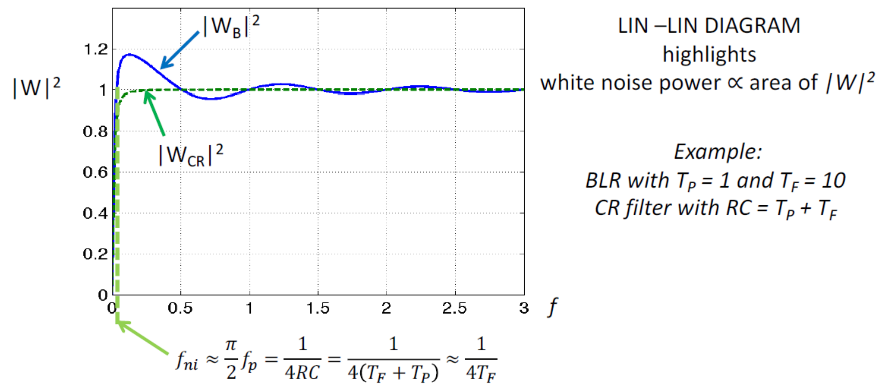


BODE DIAGRAM highlights the low-freq cutoff

Example:
BLR with T_P = 1 and T_F = 10
CR filter with RC = T_P + T_F

$f \ll 1/T_F$ (i.e. $f \ll 0{,}1$ in the example)	$1/T_F < f \ll 1/T_P$ (i.e. $f \ll 1$ in the example)
$|W_B(\omega)|^2 \approx \omega^2 (T_P + T_F)^2$	$|W_B(\omega)|^2 \approx \dfrac{\omega^2 (T_P + T_F)^2}{1+\omega^2 T_F^2}$
$|W_{CR}(\omega)|^2 \approx \omega^2 R^2 C^2$	$|W_{CR}(\omega)|^2 = \dfrac{\omega^2 R^2 C^2}{1+\omega^2 R^2 C^2}$

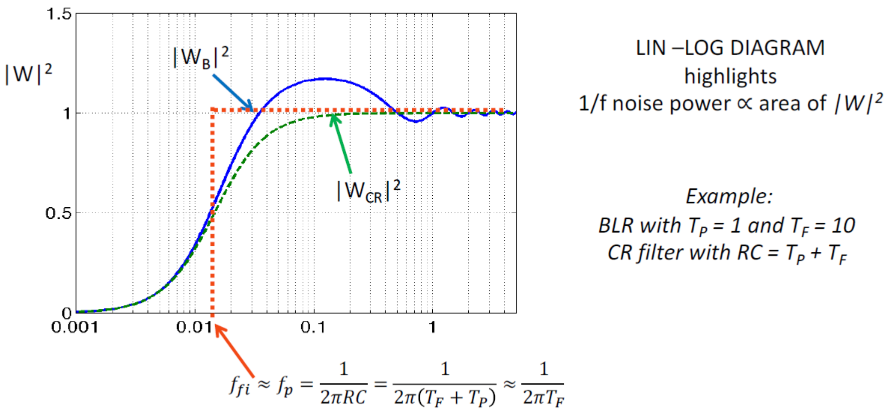
We can plot the lin-lin diagram for the WN computations.
The blue area is larger than the green due to the overshoot, but it is still much better than CDS, so we are paying less noise than CDS. The problem is the overshoot, and we need to catch how much it is big as a function of the tau. I will have a trade off between the cut of the 1/f and this overshoot.



f_{ni} = BLR high-pass band-limit for white noise. Note that:

- f_{ni} is equal to that of the equivalent CR High-pass filter
- f_{ni} is equal to bandlimit of the low-pass section in the BLR circuit

The same can be done with the 1/f noise (below).



f_{fi} = BLR high-pass band-limit for 1/f noise. Note that:

- f_{fi} is equal to that of the equivalent CR High-pass filter
- f_{fi} is equal to bandlimit of the low-pass section in the BLR circuit

The problem of the green line is that it acts also on the signal, so which one to choose? In the frequency domain can compare just the noise, I cannot change the correlation time of the noise. So we need to revert to the time domain.

SELECTION OF THE BLR PARAMETERS

To study the noise in the time domain I have to use the following parameters:

- Autocorrelation of the noise
- Autocorrelation of the w_f (to be multiplied with the one of the noise and then integral of the product).

In the time domain, there is no choice in Tp, time delay between when I open the switch and I sample it, but it must be put in correspondence of the maximum of the signal, so Tp position depends on the shape of the signal.

Then I have Tf. Tf = RC is the decay time of the LP filter part, and I can choose in general any value for it because it acts only on the noise and not on the signal, so I don't have limitations. The goal is to provide a good reduction of the 1/f noise power and to avoid the significant enhance of the WN power. In fact, e.g. with the CDS we can cut the 1/f but we double the WN. Also here I have an overshoot, so for sure we are enhancing the WN, but can we limit it?

Since at very LF the cutoff is set by 1/(Tf + Tp), it seems reasonable to reduce Tf to cut as much 1/f as possible, reducing the tau and I'm not touching the signal with the BLR. The problem of Tf = 0 is that the BLR is a delta and the reverse w_f of the LP filter. If I reduce the tau for the LP, I'm obtaining a delta and I return to CDS, and the overshoot in the BLR becomes so large that it turns into the sinusoidal waveform of the CDS.

So providing a good reduction of the 1/f noise and not enhancing the WN power are in tradeoff.

The BLR filtering is ruled by:

1. T_P **time delay** from switch opening to pulse-amplitude measurement.
   There is **no choice:** T_P is equal to the rise time from pulse onset to peak.
   In fact, T_P can't be shorter than the rise of the pulse signal and should be as short as possible for filtering effectively of the 1/f noise.
2. $T_F = RC$ **differentiation time constant: to be selected** for optimizing the overall filtering of noise. The question is: how should T_F be selected for

a) providing a good reduction of the 1/f noise power and

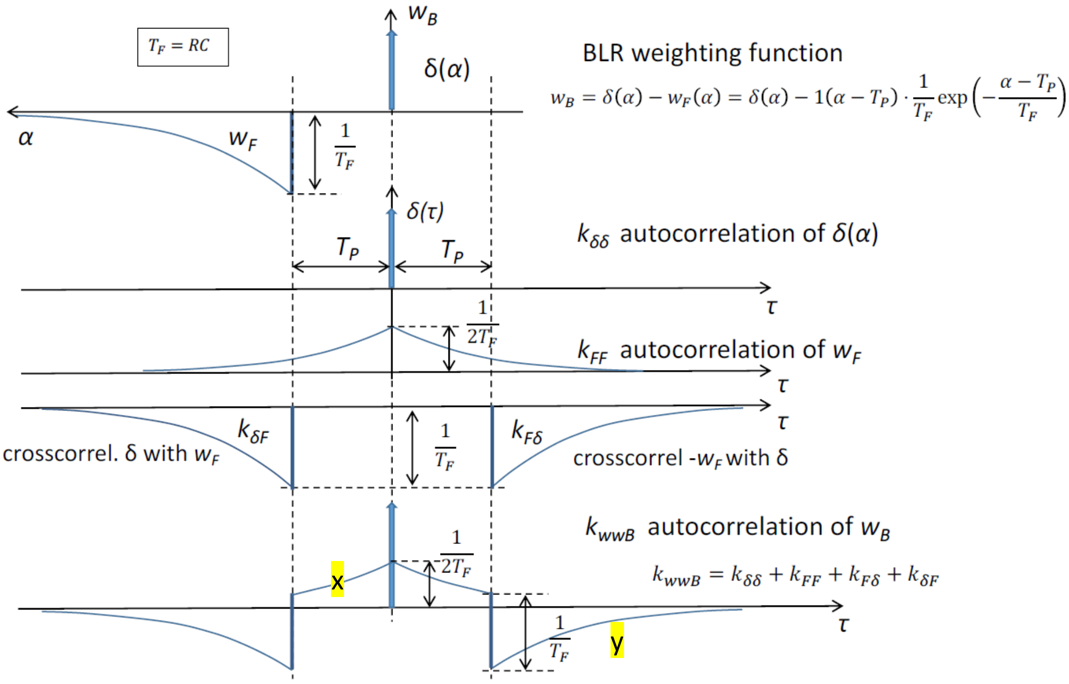
b) avoiding to enhance significantly the white noise power

Since the BLR cutoff is set by $1/(T_P + T_F)$, a very short T_F might look advisable, but it is not: a BLR with $T_F \ll T_P$ operates like a CDS, hence it doubles the white noise and remarkably enhances also the 1/f noise above the cutoff frequency.

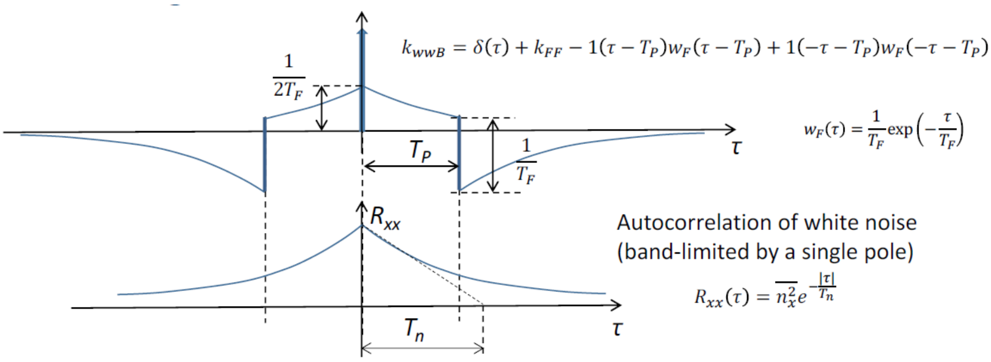
In the following discussion about the T_F selection, for focusing the ideas we will refer to a specific case: signals from a high impedance sensor processed by an approximately optimum filter, namely a CR-RC filter. The output corresponding to the input wide-band noise is a white spectrum band-limited by a simple pole. Such a situation is met in practice also in many other cases.
A better insight in the issue is gained with a **time-domain analysis of BLR filtering**

TIME DOMAIN ANALYSIS

We have to compute the autocorrelations of noise and w_f. We have the sum of a delta and a negative exponential, so the autocorrelation will be the sum of 4 terms: first squared (delta squared is a delta), second squared (negative exponential shifted squared, which gives double exponential centered in 0), cross-correlation between delta and negative exponential decay time (I get an exponential decay time in the same position) and cross-correlation between negative exponential decay time and delta (I get an exponential decay time shifted and flipped).



So the last plot is the autocorrelation, which has to be multiplied with the autocorrelation of the noise. Decay time of x is 2Tf, of y is Tf.



x $\overline{n_B^2} = \int_{-\infty}^{\infty} R_{xx}(\tau)k_{wwB}(\tau)d\tau = R_{xx}(0) + 2\int_0^\infty R_{xx}(\tau)\frac{1}{2}w_F(\tau)d\tau - 2\int_T^\infty R_{xx}(\tau)w_F(\tau - T_P)d\tau =$

$$= R_{xx}(0) + \int_{-\infty}^{\infty} R_{xx}(\tau)\frac{1}{2}w_F(|\tau|)d\tau - 2\int_0^\infty R_{xx}(\beta + T_P)w_F(\beta)d\beta$$

Denoting $r_{xx}(\tau) = \frac{R_{xx}(\tau)}{R_{xx}(0)} = \frac{R_{xx}(\tau)}{\overline{n_x^2}}$

We have

$$\overline{n_B^2} = \overline{n_x^2}\left\{1 + \int_{-\infty}^{\infty} r_{xx}(\tau)\frac{1}{2T_F}e^{-\frac{|\tau|}{T_F}}d\tau - 2\int_0^\infty r_{xx}(\beta + T_P)\frac{1}{T_F}e^{-\frac{\beta}{T_F}}d\beta\right\}$$

In the multiplication we have two parameters, Tn and Tf (Tp is fixed), but Tn might be a data, not a parameter. Tn is related to the BW of the noise; if the noise is white, Tn is really small, almost 0, but since we are trying to apply everything to a general case, the WN at the end of the procedure is the WN filtered by the matched filter, and the BW of the matched filter is the signal BW, so Tn will be related to the BW of the signal. Also Tp will be somehow related to the BW.

The good approach now is to modify the parameters to get something that I already have analyzed, since doing this product is complex. Performing the direct multiplication gives x, which is not easy to be computed.

BLR filtering of band limited WN

$$\overline{n_B^2} = \overline{n_x^2}\left\{1 + \int_{-\infty}^{\infty} r_{xx}(\tau)\cdot\frac{1}{2T_F}e^{-\frac{|\tau|}{T_F}}d\tau - 2\int_{0}^{\infty} r_{xx}(\beta + T_P)\cdot\frac{1}{T_F}e^{-\frac{\beta}{T_F}}d\beta\right\}$$

$$=$$

$$= \overline{n_x^2}\left\{1 + \frac{1}{2T_F}\int_{-\infty}^{\infty} e^{-|\tau|\left(\frac{1}{T_F}+\frac{1}{T_n}\right)}d\tau - 2e^{-\frac{T_P}{T_n}}\frac{1}{T_F}\int_{0}^{\infty} e^{-\beta\left(\frac{1}{T_F}+\frac{1}{T_n}\right)}d\beta\right\} =$$

$$= \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F} - 2e^{-\frac{T_P}{T_n}}\frac{T_n}{T_n + T_F}\right]$$

and finally

$$\boxed{\overline{n_B^2} = \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - 2e^{-\frac{T_P}{T_n}}\right)\right]}$$

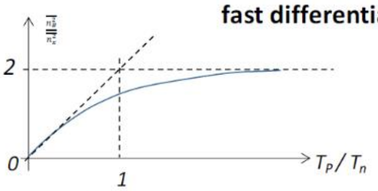
With **fast differentiation**, i.e. with $T_F \ll T_n$, it is quantitatively confirmed that the BLR acts like a CDS with $T=T_P$

$$\overline{n_B^2} \approx 2\overline{n_x^2}\cdot\left(1 - e^{-\frac{T_P}{T_n}}\right)$$

We want to use a benchmark to the test the effectiveness of the filter, and the benchmark is WN. So to have Tf really small to increase the cutoff in the frequency domain to have CDS.

The equation x we get is the equation found with CDS. It is a good thing because if Tf is reduced to 0 we have a delta for the negative part, so the same w_f of the CDS, and here we computed a formula where, if we put Tf << Tn (condition of the baseline restorer), we get exactly the baseline restorer.

fast differentiation with $T_F \ll T_n$



$$\boxed{\overline{n_B^2} \approx 2\overline{n_x^2}\cdot\left(1 - e^{-\frac{T_P}{T_n}}\right)}$$

With $T_F \ll T_n$ the effect of BLR on **band-limited white noise** depends on how long is the correlation time T_n with respect to the delay T_P

- with **short correlation time** (wide band) the noise is **doubled**:

  with $T_n < \frac{T_P}{5}$ it is $\overline{n_B^2} \approx 2\overline{n_x^2}$

- with **moderate correlation time** (moderately wide band) the noise is enhanced:

  with $T_n \approx \frac{T_P}{2}$ it is $\overline{n_B^2} \approx 1{,}73\overline{n_x^2}$

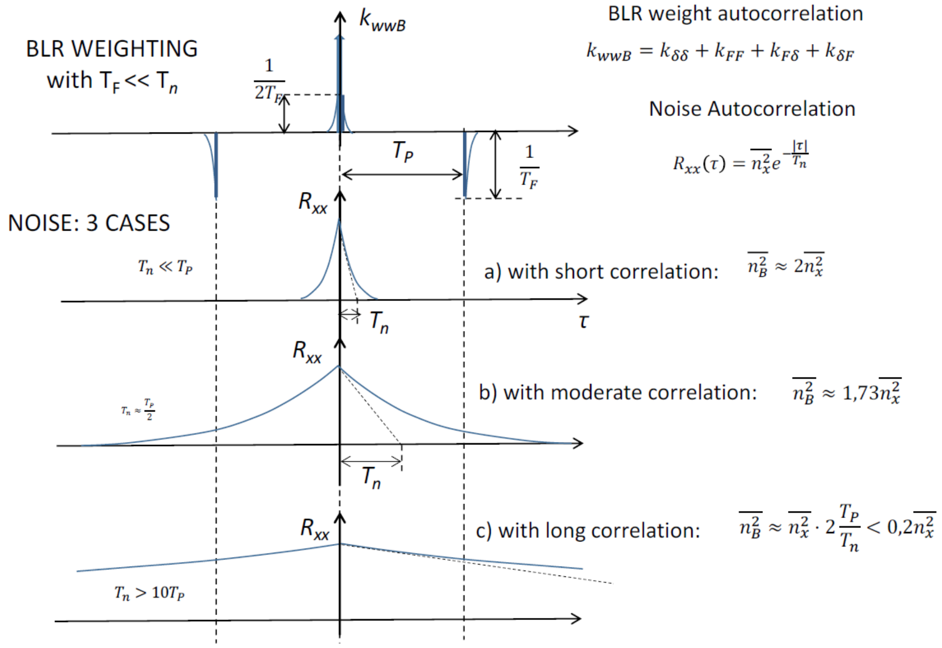
- only with **long correlation time** (low-frequency band) the noise is **attenuated***:

  with $T_n > 10T_P$ it is $\overline{n_B^2} \approx \overline{n_x^2}\cdot 2\frac{T_P}{T_n} < 0{,}2\overline{n_x^2}$

** note that anyway the level is **double** of that given by a simple CR filter with equal cutoff, that is with $T_F = RC = T_P$*

With WN, with Tn very small (short correlation time), however, we are doubling the noise. If Tn is in the same order of Tp we have an enhance of WN and if Tn is much larger than Tp (long correlation time) the WN is really attenuated. In the intermediate situation where Tn is almost Tp is the situation in output to the matched filter

Let's compute the same thing in the time domain with the autocorrelation. We plot the autocorrelation of the filter with the correct Tf and Tp.



For the noise we have 3 cases: short, moderate and long correlation times.

BLR with slow differentiation

If Tf is small we get a good CDS a good cutoff to the 1/f, because at very long correlation time the noise is strongly reduced, but we are doubling the white noise.

With T_F **NOT negligible with respect to T_n**, the effect on white noise depends also **on the size of T_F compared to T_n and T_P**. A long T_F can limit the white noise enhancement

$$\boxed{\overline{n_B^2} = \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - 2e^{-\frac{T_P}{T_n}}\right)\right]}$$

Let's evaluate how long must be T_F in the various cases of noise correlation

- with **short correlation time** $T_n \approx T_P/10$ it is

$$\overline{n_B^2} \approx \overline{n_x^2}\left(1 + \frac{T_n}{T_n + T_F}\right)$$

  for keeping $\overline{n_B^2} < 1{,}05\,\overline{n_x^2}$ we need $T_F > 20\,T_n \approx 2\,T_P$

- with **moderate correlation time** $T_n \approx T_P/2$ it is

$$\overline{n_B^2} \approx \overline{n_x^2}\left[1 + \frac{T_n}{T_n + T_F}\left(1 - \frac{2}{e^2}\right)\right] = \overline{n_x^2}\left[1 + 0{,}73\frac{T_n}{T_n + T_F}\right]$$

  for keeping $\overline{n_B^2} < 1{,}05\,\overline{n_x^2}$ in this case we need $\boldsymbol{T_F > 7T_n = 3{,}5T_P}$

Can we try to use a different Tf that doesn't touch the signal? Again we analyze the three correlation time cases, and for each we try to choose Tf.
For short correlation time, to keep the output noise equal to the input noise times 50%, so 1.05 of the output, we need Tf > 20Tn to limit the enhance of the WN. However, this is not a so common situation.

The real important situation is in the case of moderate correlation time, with Tn in the same order of magnitude of Tp. Also in this case an enhance of 1.05 of the original noise. Tf muste be 3.5*Tp.

For long correlation times, the noise is attenuated just for any Tf, like in the CR.

- with **long correlation time** $T_n > 10\ T_P$ it is

$$\overline{n_B^2} \approx \overline{n_x^2}\left[1-\frac{T_n}{T_n+T_F}\right]=\overline{n_x^2}\frac{T_F}{T_n+T_F}$$

  No problem with such a low-frequency noise: it is attenuted by the BLR just as by a CR constant-parameter filter (with equal time constant $T_F = RC$)

The most interesting case for us is noise with moderate T_n . In fact, when the BLR works on the output of an optimum (or approximate-optimum) filter for wideband noise, the correlation time T_n and delay T_p are comparable, since they are both closely related to the band-limit of the signal pulse.

- We conclude that for avoiding enhancement of the white noise it is necessary to select a fairly slow BLR differentiation, i.e. a fairly long T_F

$$\boxed{T_F \geq 5T_P}$$

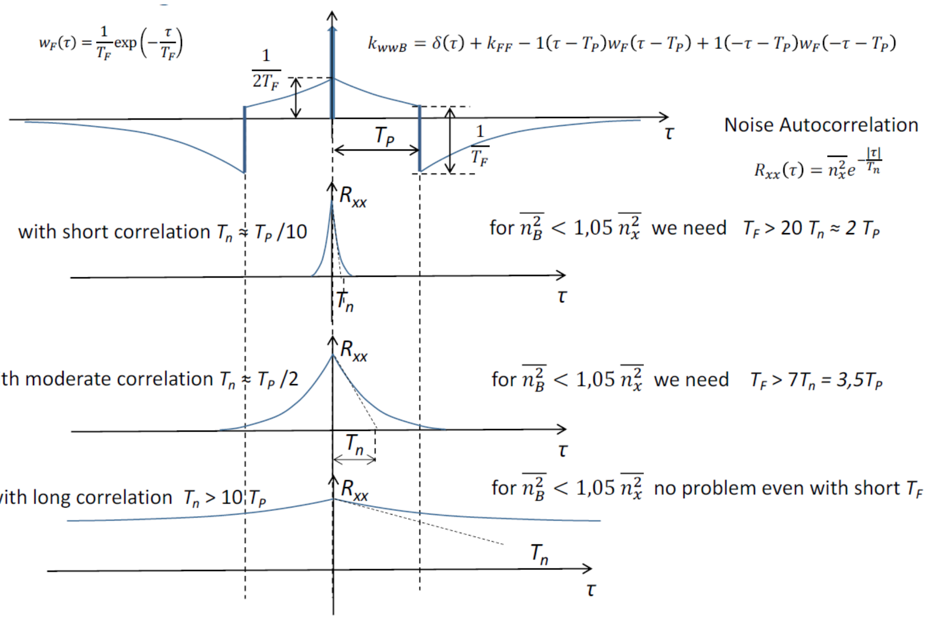
- This approach is satisfactory also for filtering the 1/f noise, notwithstanding that making T_F longer than T_P shifts down the BLR cutoff frequency, hence reduces the attenuation of 1/f noise. This is counterbalanced by the fact that the enhancement of 1/f noise at frequencies above the cutoff is limited by the low-pass filtering in the baseline subtraction, whereas with short T_F it is remarkable.

In the end, we can get good results with attenuation of 1/f and small enhancement of WN for Tf ~ 5*Tp.

As soon as we increase Tf we are reducing the enhancement of the noise and with Tf = 5*Tp we can limit the enhancement to 5% but we are also moving the cutoff in the frequency domain, so increasing the 1/f.

In a situation where the 1/f is orders of magnitude stronger than the WN, probably is better to cut completely the 1/f noise also if we are doubling the WN. Otherwise, if the 1/f is not so big, it is important to remove it without enhancing the noise we already have.

BLR filtering with slow differentiation



Summary

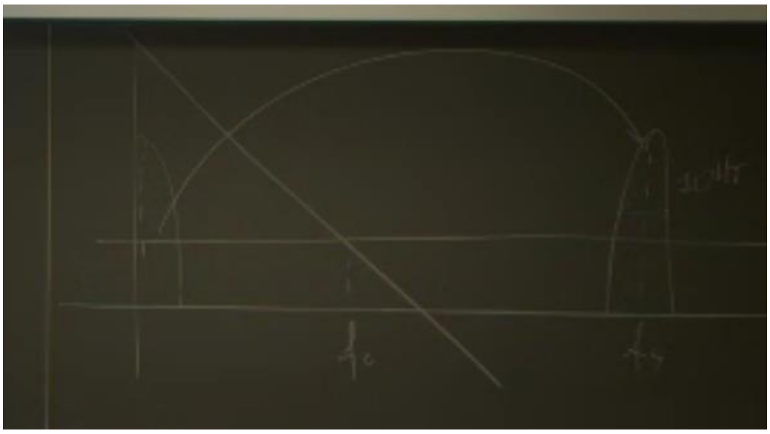
- The BLR is a high-pass filter that acts on noise and disturbances without affecting the pulse signal. This is the good thing.
- The BLR is a switched-parameter filter; the low-pass section within the high-pass filter structure is a boxcar integrator that acquires the baseline only in the intervals free from pulses.
- The BLR can thus establish a high-pass band-limit at a high value (suitable for reducing efficiently the 1/f noise output power) without causing the signal loss suffered with a constant-parameter high-pass filter having the same band-limit. So **we can change the Tf without affecting the signal**.
- The high-pass band-limit enforced by the BLR is given (with good approximation) by the low-pass bandlimit of the low-pass section in the BLR circuit structure. Since normally Tp is small, if we change Tf the BW of the LP filter gives also the cutoff for the 1/f.
- The combination of: (1) optimum filter designed for the case of pulse signal in presence of wideband noise only (i.e. without 1/f noise) and (2) BLR specifically designed (for reducing the actual 1/f noise without worsening the wide-band noise) provides in most cases a quasi-optimum filtering solution.

BANDPASS FILTERS

In the frequency domain, can we move the signal in a region where there is no 1/f noise? If I can move the signal at HF where there is not the 1/f noise, I have solved the problem of 1/f. This is not always possible, however. If I can move the signal, do I also improve the SNR?

NARROW BAND SIGNAL

Someone is giving us a narrow-band signal, whose band is very small with a center frequency fs. Narrow band means BW < 10 Hz, and the BW is much smaller than fs, so that we can clearly see the signal in the frequency domain → something similar to a truncated sinusoidal waveform.



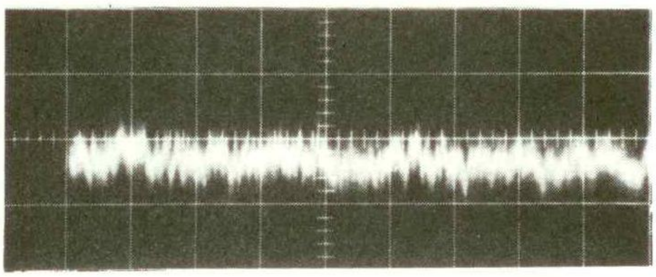
Can we measure such narrow base signal in presence of WN and 1/f noise? Normally the situation is in the image below, with a lot of noise superimposed to the signal.

Power signals with a narrow power spectrum, that is, a peak with

- center-frequency f_s
- bandwidth Δf_s which is small in absolute value, typically $\Delta f_s < 10$ Hz, and/or with respect to the center frequency $\Delta f_s << f_s$

They approximate well a sinusoid over a wide time interval $T_s \approx 1/\Delta f_s$

QUESTION: how can we measure such narrow-band signals in presence of intense white noise? And what if also 1/f noise is present?



RECOVERING NARROW-BAND SIGNALS FROM NOISE

The specs are the one in the image below. Then we create the setup with the amplifier, and so we will have an upper limit, e.g. 1 MHz. The WN PSD is 5nV/sqrt(Hz), and for the 1/f we take fc = 2 kHz (standard frequency corner).

We want to study what happens in 3 different cases:

1. HF, with fs = 100 kHz, so very far from the frequency corner.
2. fs = 1 kHz, just below the frequency corner.
3. fs = 10 Hz, so exactly in the middle of 1/f

Let's see some typical examples of signals with

- narrow linewidth $\Delta f_s = 1\ Hz$
- small amplitude $V_s \leq 100\ nV$

for bringing them to higher level (suitable for processing circuits: filters, meters, etc.) they are amplified by a DC-coupled wide-band preamplifier with

- upper band-limit $f_h = 1MHz$
- noise spectral density (referred to input) with «white» component $\sqrt{S_b} = 5nV/\sqrt{Hz}$ and *1/f* component with corner frequency $f_c = 2kHz$

Let us consider three cases with different center-frequency f_s :

- Case 1: **high** frequency f_s = 100 kHz
- Case 2: **moderately low** frequency f_s = 1 kHz
- Case 3: **low** frequency f_s = 10 Hz

Case 1

The signal is at 100 kHz, and it's really hard to detect the signal in the oscilloscope. For the signal we are at 100 kHz, and fc of 1/f is 2 kHz, so I can cut the 1/f placing a HP filter at e.g. 10 kHz, so that I don't compromise the signal and I'm after the fc. If we cut all the 1/f, the only noise that remains is the WN, whose value is the PSD times the BW, where fh is the BW of the amplifier or of the subsequent LP filter.

CASE 1: signal $V_s \leq 100\ nV$ at high frequency f_s = 100 kHz

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

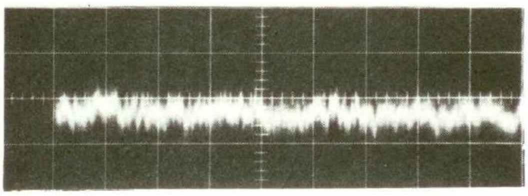
The signal to be recovered is at frequency f_s = 100 kHz much higher than the noise corner frequency $f_c = 2kHz$, so that we can use a simple high-pass filter with band-limit f_i = 10kHz to cut off the 1/f noise and obtain a rms noise (referred to the preamp input)

$$\sqrt{\overline{v_n^2}} = \sqrt{S_b} \cdot \sqrt{(f_h - f_i)} \approx \sqrt{S_b} \cdot \sqrt{f_h} = 5\mu V$$

and therefore

$$\frac{S}{N} = \frac{V_S}{\sqrt{\overline{v_n^2}}} \leq 0{,}02 << 1$$

Even the highest signal $V_s = 100\ nV$ is **practically invisible on the oscilloscope display**! The noise covers a band $\approx 5 \times rms\ value \approx 20\ \mu V$ and the sinusoidal signal is buried in it!



Vertical scale 50µV/div

Horizontal scale 5µs/div

The problem is that if we now compute the SNR, we get 100nV/5uV and the SNR is 0.02, much lower than 1. This is not strange, because if we look at the oscilloscope, we are not able to distinguish the signal. So from the time domain standpoint we cannot distinguish the signal so far.

If we move to the frequency domain, with a spectrometer we can detect the signal. The point is that, why can I detect the signal in the frequency domain with the spectrometer but not in the time domain or with the oscilloscope. The problem is that in the time domain, for noise computations, we are acquiring all the possible WN and then comparing it to the signal. With the spectrometer, or in the frequency domain, we are instead comparing the amplitude of the WN and of the signal, we are not interested in the integral of the signal and of the noise as in the time domain. We have to translate this idea in the time domain.

We can compute the power of the signal, which is included in 1 Hz, so in the frequency domain we can approximate the signal with a rect with an amplitude of 1 Hz. The result is 70 nV/sqrt(Hz). This because the integral of the power spectrum is the power.

Then we need to understand the power of the noise, which is 5 nV/sqrt(Hz).

CASE 1: signal $V_s \leq 100\ nV$ at high frequency f_s = 100 kHz

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

SIGNAL: the power $P_S = \frac{V_S^2}{2} = 50 \cdot 10^{-16}\ V^2$ is within a bandwidth $\Delta f_S = 1\ Hz$

so that the effective power density of the signal is $\sqrt{S_S} \approx \sqrt{\frac{P_S}{\Delta f_S}} = \frac{70nV}{\sqrt{Hz}}$

NOISE: the effective power density at f_s = 100 kHz is $\sqrt{S_b} = 5nV/\sqrt{Hz}$

On the spectrum analyzer display the signal peak is **very well visible above the noise!**

$$\frac{\sqrt{S_S}}{\sqrt{S_b}} = 14 \gg 1$$

Conclusion: good S/N can be obtained with a bandpass filter having bandwidth Δf_b matched to the signal band $\Delta f_b \approx \Delta f_s$

$$\frac{S}{N} = \sqrt{\frac{P_S}{S_b \Delta f_b}} = \sqrt{\frac{S_S \Delta f_S}{S_b \Delta f_b}} \approx \frac{\sqrt{S_S}}{\sqrt{S_b}} = 14 \gg 1$$

But how can we get this situation? The idea is to use a BP filter with a BW that is equal to the one of the signal. Thus, we are integrating the power spectrum of 1 Hz of the signal, so all the signal, and 1Hz of the noise. When creating a BP filter, if the Q factor is huge, theoretically it is not possible to implement a BP filter. Even so, the filter would be so unstable to be practically unusable. In this case the Q = 10^5.

Case 2

We do the same computations of case 1, but we cannot cut the 1/f as before. For sure we will include a HP filter, but since the signal is at 1 kHz and we don't want to touch it, we place the pole at 100 Hz, one decade before → some 1/f noise is included.

The good thing is that the 1/f noise part is negligible because depending on the ln, and the WN is instead integrated over almost all the frequencies after the HP filter pole.
In the end, as before, we cannot detect the signal in the time domain.

CASE 2: signal $V_s \leq 100\ nV$ at moderately low frequency f_s = 1 kHz

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at f_s = 1 kHz just below the corner frequency $f_c = 2kHz$.
For reducing the 1/f noise we can still use a high-pass filter, but in order to pass the signal the band-limit f_i must be reduced: $f_i << f_s$ = 1 kHz, typically f_i = 100Hz.
The rms noise referred to the input is

$$\sqrt{\overline{v_n^2}} \approx \sqrt{S_b(f_h - f_i) + S_b f_c \ln\left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h + S_b f_c \ln\left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h} \approx 5\mu V$$

1/f noise is negligible $\quad S_b f_c \ln\left(\frac{f_h}{f_i}\right) \ll S_b f_h$

and therefore

$$\frac{S}{N} = \frac{V_S}{\sqrt{\overline{v_n^2}}} \leq 0{,}02 << 1$$

The situation is practically equal to that of Case 1: the signal is **practically invisible on the oscilloscope display, it's buried in the noise**!

So the thing to do is to move to the frequency domain.

CASE 2: signal $V_s \leq 100\ nV$ at moderately low frequency f_s = 1 kHz

b) observing the power spectrum in FREQUENCY DOMAIN, i.e. on spectrum analyzer display

SIGNAL: the power $P_S = \frac{V_S^2}{2} = 50 \cdot 10^{-16}\ V^2$ is within a bandwidth $\Delta f_S = 1\ Hz$

so that the effective power density of the signal is $\sqrt{S_S} \approx \sqrt{\frac{P_S}{\Delta f_S}} = \frac{70nV}{\sqrt{Hz}}$

NOISE: due to the **1/f noise**, the effective power density at f_s = 1 kHz is somewhat higher

$$\sqrt{S_n(f_s)} = \sqrt{S_b + S_b\frac{f_c}{f_s}} = \sqrt{S_b}\sqrt{1 + \frac{f_c}{f_s}} = \sqrt{3} \cdot \sqrt{S_b} \approx 8{,}7\frac{nV}{\sqrt{Hz}}$$

Anyway, on the spectrum analyzer display the signal peak is still **well visible above the noise**

$$\frac{\sqrt{S_S}}{\sqrt{S_n(f_S)}} = 8 > 1$$

Conclusion: a bandpass filter with bandwidth Δf_b matched to the signal $\Delta f_b \approx \Delta f_s$ still gives a **fairly good S/N**

$$\frac{S}{N} = \sqrt{\frac{P_S}{S_n(f_S)\Delta f_b}} = \sqrt{\frac{S_S \Delta f_S}{S_n(f_S)\Delta f_b}} \approx \frac{\sqrt{S_S}}{\sqrt{S_n(f_S)}} = 8 > 1$$

In case 1 we compared the signal with the WN. Now we cannot compare the signal with just the WN, because we have also the 1/f. However, since the BW is small, we can compare the 70 nV with the WN and the value of the 1/f noise in the central frequency of the signal.

The contribution of the 1/f is not so high, the SNR is reduced but still 8, quite good. So if we could make a BP filter with 1Hz, we could get a SNR = 8. Now the Q = 1000, so not huge as before, but also in this case we cannot implement the BP. So also in this case 2 we can detect a signal with a filter that doesn't exist, but ok, we can detect the signal.

Case 3

CASE 3: signal $V_s \leq 100\ nV$ at low frequency $f_s = 10$ Hz

a) observing the voltage waveform in the TIME DOMAIN, i.e. on oscilloscope display

The signal is now at f_s = 10 Hz much below the corner frequency $f_c = 2kHz$.
For reducing the the 1/f noise we can still use a high-pass filter, but with strongly reduced band-limit $f_i << f_s$ = 10 Hz, typically f_i = 1 Hz. The rms noise referred to input is

$$\sqrt{\overline{v_n^2}} \approx \sqrt{S_b(f_h - f_i) + S_b f_c \ln\left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h + S_b f_c \ln\left(\frac{f_h}{f_i}\right)} \approx \sqrt{S_b f_h} \approx 5\mu V$$

1/f noise is negligible $\quad S_b f_c \ln\left(\frac{f_h}{f_i}\right) \ll S_b f_h$

and therefore $\quad \frac{S}{N} = \frac{V_S}{\sqrt{\overline{v_n^2}}} \leq 0{,}02 << 1$

The situation is practically equal to that of Case 1 : the signal is **practically invisible on the oscilloscope display, it's buried in the noise**!

Also in this case the 1/f noise is negligible with respect to the WN. So in the end we are increasing of 2 orders of magnitude the frequency and the 1/f is inside the log so it doesn't change too much, so it is negligible comparable with all the integral up to 1 MHz.
The computations for the signal are the same as before.
As for the noise, we have to do the same computation of case 2. We get a signal that is the same level of the noise even if we use a BP with a BW equal to the one of the signal.
So the 1/f is a problem or not depending on the frequency. If we are further or close to the fc and we can create the BP filter we are ok, but if we are in the middle of 1/f noise there is no way to recover the signal.

In the end, if I can move the signal, better to move it at least one decade of fc, as in case 1.

Summary

- For a narrow-band signal plunged in white noise (i.e. with frequency fs higher than the 1/f noise corner frequency fc) a bandpass filter matched to the signal band is very efficient and makes possible to recover signals even so small that they are buried in the wide-band noise.
- For a narrow-band signal plunged in dominant 1/f noise (i.e. with fs lower than the 1/f noise corner frequency fc) a bandpass filter matched to the signal is still quite efficient and in many cases makes possible to recover the signal. However, if we consider signals at progressively lower frequency fs, the 1/f noise density at fs progressively rises, so that the available S/N is progressively reduced.

Open questions

- We need efficient band-pass filters with very narrow band-width. We need to understand how to design and implement such narrow-band filters , but we shall deal with this issue after dealing with the following question.

- If the information is carried by the amplitude of a low-frequency signal, it has to face also 1/f noise. **It would be advantageous to escape this noise by preliminarily transferring the information to a signal at higher frequency**. However:
  - How can we transfer the signal to higher frequency if I'm able to create the BP filter?
  - If we transfer to the higher frequency also the 1/f noise that faces the signal, this makes the transfer useless: how can we avoid it? How can I move only the signal and not the 1/f

noise? This is normally an issue. In fact, if e.g. we move the signal with something (multiplier) that has a lot of WN or 1/f noise, we are again in trouble.

- For escaping 1/f noise, a low-frequency signal should be transferred to higher frequency before it mixes with 1/f noise of comparable power density: that is, frequency transfer should be done before the stage where the signal meets the 1/f noise source.

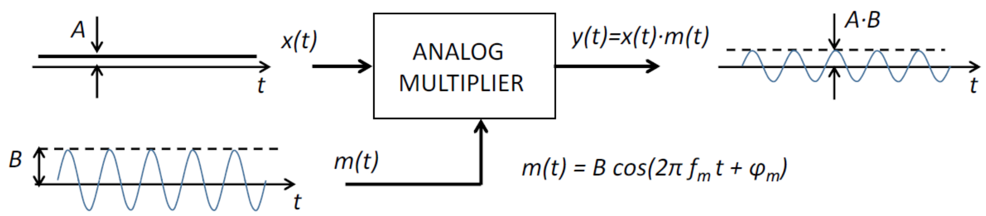
- The frequency-transfer stages have their own noise, with different intensity in different types. Unluckily, the types with lowest noise bear other drawbacks, typically a limited capability of transfer, restricted to moderately high frequency.

- For achieving our goal, the signal must be higher than the noise referred to the input of the frequency-transfer stage. If with a given stage the signal is not high enough, preamplifying is not advisable because a preamp brings its 1/f noise. In most cases it is better to transfer the signal «as it is» by means of a frequency transfer stage with lower noise and accept the limitations of this stage, typically a moderate operating frequency.

MOVING SIGNALS IN FREQUENCY – SIGNAL MODULATION

At this moment we suppose that we can create any BP filter, and we focus on moving the signal. The idea is the modulation.

Let's consider a small in BW signal centered around zero with small amplitude, and we are interested in the amplitude. The idea of modulation is to multiply the signal times a reference, that in this case is a cosinusoidal waveform. Then we multiply the two in the time domain. We get a cosinusoid with amplitude A*B, and B is a data.



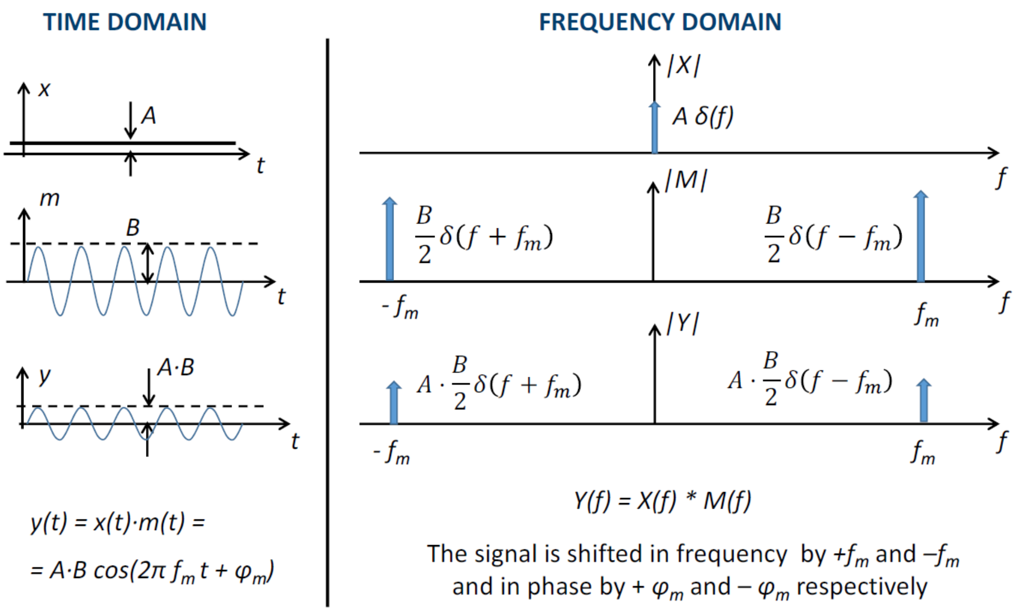
- **Information is brought by** the (VARIABLE) amplitude A of a DC signal *x(t) = A* .
  (NB: a real DC signal is a signal at very low frequency with very narrow bandwidth)
- An analog multiplier circuit combines the signal with a sinusoidal waveform *m(t)* *(called reference or carrier)* with frequency f_m and CONSTANT amplitude *B*
- The information is transferred to the amplitude of a sinusoidal signal *y(t)* at frequency f_m

$$y(t) = A \cdot B \cos(2\pi f_m t + \varphi_m)$$

Normally phi_m disappears later on, because phi_m = 0 is a value that allows to maximize the SNR.

What we want to do is a multiplication in the time domain, so a convolution in the frequency domain. A constant value in the time domain is a delta in the frequency domain, something that is almost constant in the time domain, it is something with a small BW in the frequency domain. In the frequency domain the cosine has two deltas at fm and -fm, with a positive sign.

So let's move to the frequency domain, where we need to make the convolution.



We have a problem. In fact, the result of the convolution is not always correct.

Convolution in the frequency domain

In the **time domain (TD)** the amplitude modulation is the **multiplication** of the signal *x(t)* (with variable amplitude A) by the reference waveform *m(t)* (with standard amplitude B)

$$y(t) = x(t) \cdot m(t)$$

In the **frequency domain (FD)** it is the **convolution** of the transformed signal *X(f)* by the transformed reference M*(f)*

$$Y(f) = X(f) * M(f) = \int_{-\infty}^{\infty} X(\alpha)\, M(f-\alpha)\, d\alpha \quad x$$

Convolution is more complicated in FD than in TD because:

1. the functions to be convolved are twofold, that is, they run in the positive and negative sense of the frequency axis
2. **Complex** values must be summed at every frequency for obtaining *Y(f)*.

In general the result of FD convolution is not as intuitive as that of TD convolution and the module |Y(f)|is **NOT** given by the convolution of |X(f)| and|M(f)|

$$|Y(f)| \neq |X(f)| * |M(f)|$$

we must first compute the real and imaginary parts of Y(f) and then obtain |Y(f)|

Convolution is more complex in the frequency domain rather than in the time domain. In fact, in the time domain we don't have 'negative time', and we don't have the imaginary axis, in the time domain we have real functions. In general, any Fourier transform in the frequency domain is a complex number. So integral x should be done with complex number, not trivial.

In general, **the convolution in the frequency domain is not given by the convolution of the absolute value of the two signals**. The problem is that, from a graphical standpoint, we can make only the convolution of the amplitudes, not the convolution of the phase.

However, there are some special cases where the convolution, at first approximation, is the convolution of the modulus. These cases are:

1. X(f) confined in a narrow BW, and this is the case of our signal.

2. M(f), the reference function, has a spectrum composed just by deltas with a fundamental fm is much greater than the signal BW. This because we don't want the overlapping of the replicas, otherwise we would need to convolve also the phases.

If both conditions 1 and 2 are verified, we can make the convolution of the modulus, shifting X(f) on any line (delta) of the second function M(f). We still get a complex function, but replicas are not overlapping.

In the cases here considered, however, the issue is remarkably simplified because

a) *X(f)* is confined in a **narrow** bandwidth Δf_s

b) *M(f)* has a line spectrum with (fundamental) frequency f_m that is much greater than the signal bandwidth $\boldsymbol{f_m >> \Delta f_s}$

In the convolution *X(f) * M(f)* each line of M(f) acts on *X(f)* as follows

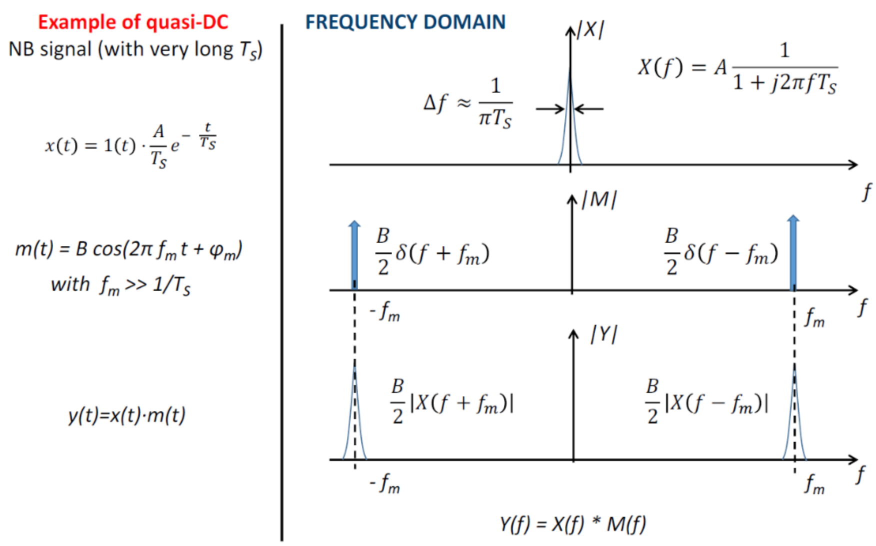
- Shifts in frequency every component of *X(f)* by $+\boldsymbol{f_m}$ and $\boldsymbol{-f_m}$ (i.e. adds to each frequency $+\boldsymbol{f_m}$ and $\boldsymbol{-f_m}$)
- Shifts in phase every component of *X(f)* by $+\varphi_m$ and $-\varphi_m$ (i.e. adds to every phase $+\varphi_m$ and $-\varphi_m$)

*In cases with $\Delta f_s << f_m$, there is **no sum of complex numbers** to be computed because at any frequency f there is at most one term to be considered, all other terms are negligible.*

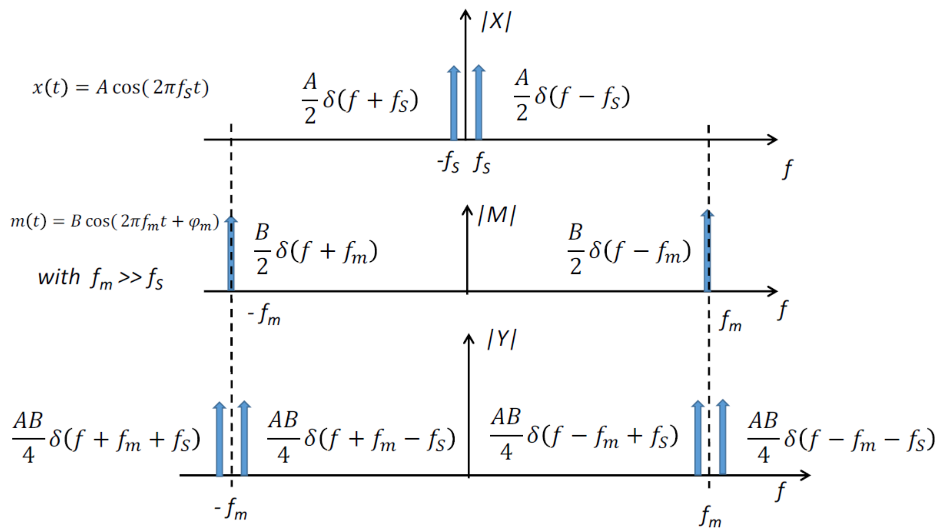
The result of the convolution is easily visualized: every line of *M(f)* shifts *X(f)* in frequency and adds to *X(f)* its phase. Therefore, |Y(f)| is well approximated by the convolution of |X(f)| and |M(f)| and $|Y(f)|^2$ by the convolution of $|X(f)|^2$ and $|M(f)|^2$

$$\boxed{|Y(f)| \cong |X(f)| * |M(f)|} \qquad \boxed{|Y(f)|^2 \cong |X(f)|^2 * |M(f)|^2}$$

Example



Now we want to apply this idea to a signal that is also a cosinusoid, as below. Also the reference is a cosinusoidal. Delta_fs of the signal is 2*fs = BW, and it must be much smaller than fm, so the hypothesis is verified.



Now we want to check the same things in the time domain, moving to it.

By exploiting the a well known trigonometric equation

$$\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha - \beta) + \frac{1}{2}\cos(\alpha + \beta)$$

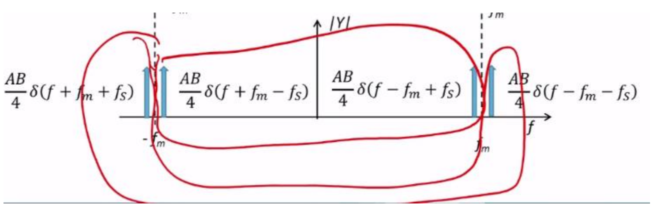
in cases with sinusoidal signal and sinusoidal reference

$$x(t) = A\cos(2\pi f_S t)$$
$$m(t) = B\cos(2\pi f_m t + \varphi_m)$$

the result is directly obtained

$$\boxed{y(t) = x(t)\cdot m(t) = \frac{AB}{2}\cos[2\pi(f_S - f_m)t - \varphi_m] + \frac{AB}{2}\cos[2\pi(f_S + f_m)t + \varphi_m]}$$

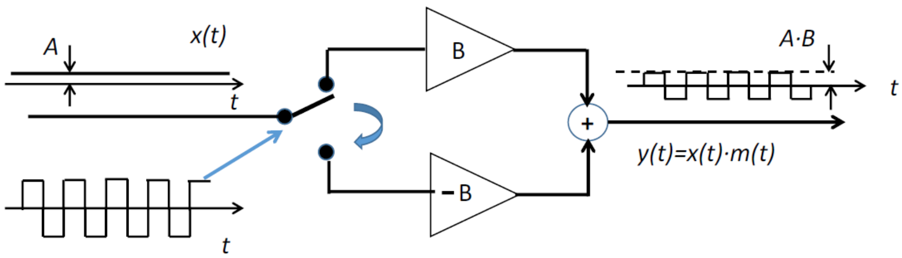
We get the final formula in the box. The frequency of the cosine is fs – fm and the other fs + fm. It makes sense because the external deltas are centered at fs + fm, and the inner ones at fs – fm.



These are the cosinusoids in the frequency domain.

SQUAREWAVE AMPLITUDE MODULATION

Modulation with a squarewave reference *m(t)* can be implemented with circuits based simply on switches and amplifiers, without analog multipliers

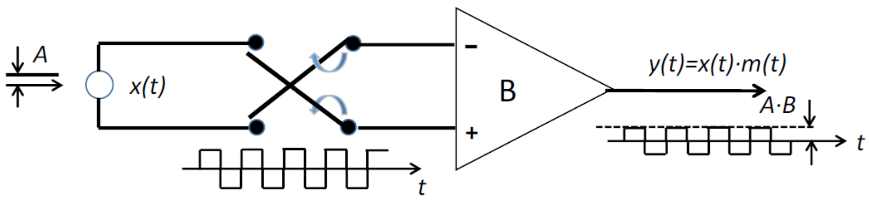


- In such cases, the circuit noise referred to the input is due mostly to the switch-devices and is much lower than that of analog multiplier circuits.
- **Metal-contact switches** have the lowest noise, but they can operate at limited switching frequency, typically up to a few 100Hz
- **Electronic switches** (MOSFET, diodes, etc.) operate up to very high frequencies but have fairly higher noise (anyway MOSFETs operating as switch-device have lower noise than MOSFETs operating as amplifier devices).

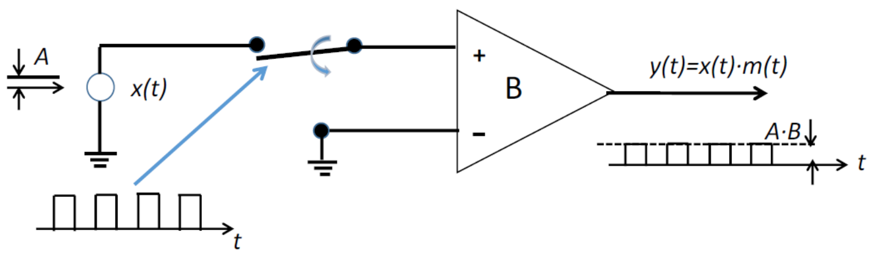
It is not a sinusoidal waveform, but it is a much easier modulation than with an analog approach, because treating switches is easier. With a square wave modulation we can also avoid to double the noise. Furthermore, the cost is much smaller.

We could also use just one amplifier because we need to create +B for the signal and -B for the reference. If +B is different from -B, we get an offset, which in the frequency domain is a delta in 0. Then when I'm convolving in the frequency domain, if I have a line at 0 Hz, I'm moving the signal also at 0 Hz, not only at the frequencies of the reference. The problem is that we have 1/f noise at 0 Hz, so we are placing part of the signal in the 1/f noise zone. So we use a single amplifier to have the same B for + and -.

Switching example: differential amplifier with alternated input polarity



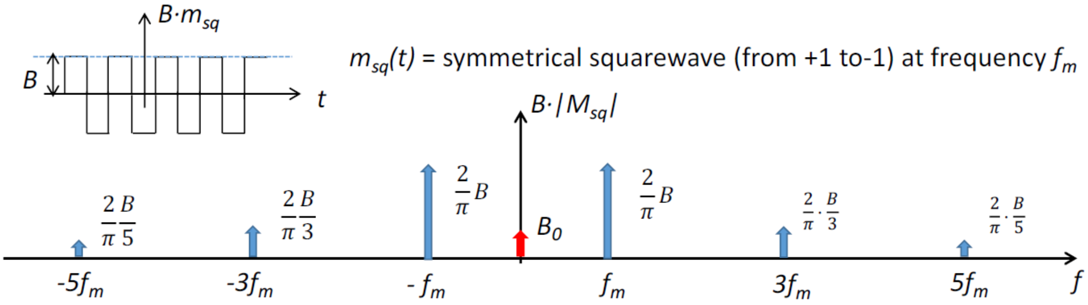
Switching example: chopper (ON-OFF modulation)



Sometimes, we could switch from amplifier and ground (second plot), but we have to manage the offset, because the ground will have an offset. We could use this solution because sometimes we are forced to use it because for instance the modulation is not an electrical modulation but an optical one, e.g. the sensor is a light signal and we want to modulate the light. Instead, it is quite difficult to create 'negative light', so to modulate a light signal the second approach of the previous image is the only way to go.

In both cases, we have to manage square waveforms.

Squarewaves and F-transforms.



$$\boxed{B \cdot M_{sq}(f) = B \cdot \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2}\left[\delta(f - f_{2k+1}) + \delta(f + f_{2k+1})\right]}$$

with $\begin{cases} f_{2k+1} = (2k+1) f_m \\ b_{2k+1} = \dfrac{(-1)^k}{(2k+1)} \cdot \dfrac{4}{\pi} \end{cases}$

In the amplitude modulation:

- each line of the reference M acts like a simple sinusoidal reference, i.e. shifts by its frequency and its phase the signal X and multiplies it by the amplitude $B \cdot b_{2k+1}$
- if the squarewave is not perfectly symmetrical (e.g. it has asymmetrical amplitude and/or duration of positive and negative parts) there is also a finite DC component with amplitude B_0 (possibly very small)
- the DC component does NOT transfer the signal X in frequency, just «amplifies» it by B_0

The Fourier transform of a square waveform is a sum of deltas. How can we get this Fourier transform? The good thing of the transform is that it is composed only by lines, and it is a requirement for the reference. The red arrow is the small delta at 0 Hz that is present if we have a small offset in the B values, and we want to avoid it.

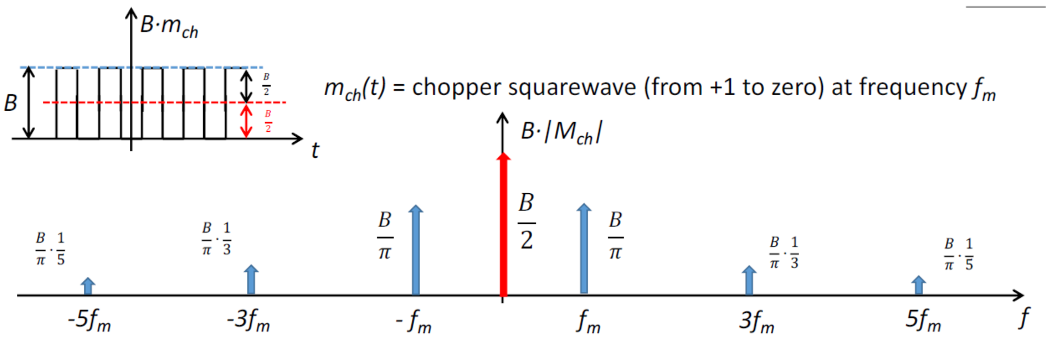
We can study the chopper squarewave. We can study it as a symmetric square waveform plus an offset of B/2. And the offset due to B/2 is a delta at 0Hz, so it is the exact same Fourier transform of the previous case but with a delta in the middle.

Chopper waveform

Nowadays, all signals are fast and optical. When we have an optical signal and we want to modulate it, we cannot use just a sinusoidal modulation, because the light cannot be negative. So to create a square waveform that is 'light' and 'no light', we can use a wheel with holes that is rotating, so that the laser is passed either through the hole or stopped. If we increase the number of holes we are changing the duty cycle. This is the chopper.

The chopper waveform helps because a replicated signal in the time domain corresponds to sampling in the frequency domain. So to compute the Fourier transform of something that is replicated, I take the Fourier of one replica (the sinc for the rect) and then I sample it, thus getting the deltas of the Fourier transform in the next image. Of course we need to take the modulus.

Also in this case we are happy because we have only lines. The only problem is the red delta at 0Hz.



Chopper squarewave with amplitude B =

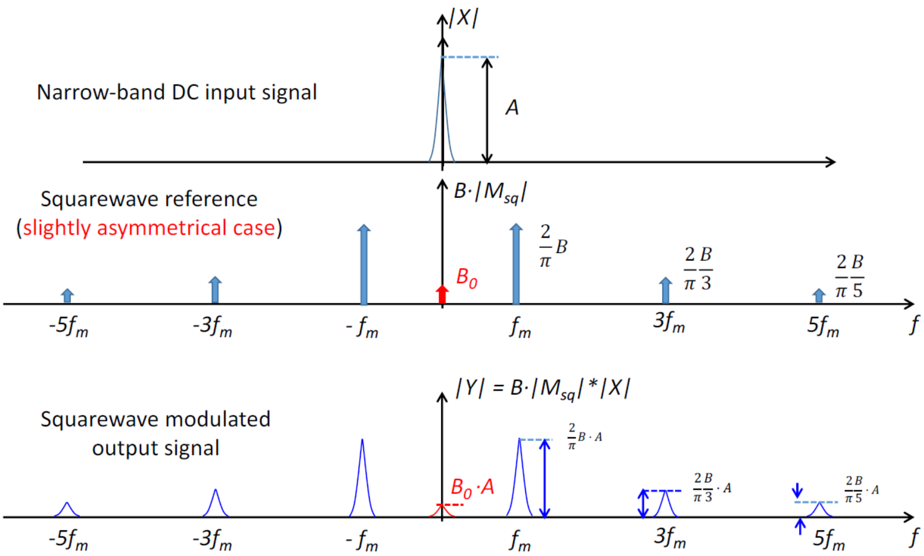
= Symmetrical squarewave with amplitude B/2 + DC component with amplitude B/2

$$\boxed{B \cdot M_{ch}(f) = \frac{B}{2} \cdot M_{sq}(f) + \frac{B}{2} \cdot \delta(f)}$$

In the amplitude modulation by a chopper:

- a replica of the signal X «amplified» by B/2 is transferred in frequency by the squarewave
- another replica of X «amplified» by B/2 is NOT transferred, it stays where it is

If we have the signal, we need to check if its BW is much smaller than the distance of the lines, fm. Then if the reference is composed only by lines. At this point we can perform the convolution in a graphical way as below.



The problem is that we are increasing the number of problems we have, because the signal is no more cosinusoidal, but a sum of BP filters that doesn't exist.

Summary

- As intuitive, narrow-band filtering is very effective for recovering narrow-band signals immersed in wide-band noise.
- Besides wide-band noise, however, other components with power density increasing as the inverse frequency (1/f noise) are ubiquitous in electronic circuitry (amplifiers etc.). In the low-frequency range they are indeed dominant.
- At low frequencies the 1/f noise added by the circuitry is overwhelming, so that the solution of narrow-band filtering becomes progressively less effective and finally insufficient for recovering signals with progressively lower frequency.

- An effective approach to recover a low-frequency signal is to move it to higher frequency before the addition of 1/f noise. That is, to modulate the signal before the circuitry that contains the 1/f noise sources.
- Narrow-band filtering can then be employed to recover the modulated signal; we will now proceed to analyze methods and circuits for narrow band filtering.

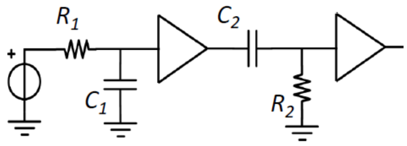
BP FILTERING WITH HP PLUS LP FILTERS – CR/RC

The basic idea to develop a BP filter is to use a LP and HP filter combined. In the middle we need to put a buffer to decouple the impedance. We are interested in a very small BW, so I suppose that the same value for the pole of the HP and LP is chosen, they have the same tau.

Cascaded two-cell filter:

low-pass $T_1 = R_1C_1$ $f_{p1} = 1/2\pi T_1$

high-pass $T_2 = R_2C_2$ $f_{p2} = 1/2\pi T_2$



$$H = \frac{V_{out}}{V_{in}} = \frac{1}{1 + \frac{jf}{f_{p1}}} \cdot \frac{\frac{jf}{f_{p2}}}{1 + \frac{jf}{f_{p2}}} \qquad |H|^2 = \frac{1}{1 + \left(\frac{f}{f_{p1}}\right)^2} \cdot \frac{\left(\frac{f}{f_{p2}}\right)^2}{1 + \left(\frac{f}{f_{p2}}\right)^2}$$

With equal poles $T_1 = T_2 = T$ and $f_{p1} = f_{p1} = f_p$

$$H = \frac{\frac{jf}{f_p}}{\left(1 + \frac{jf}{f_p}\right)^2} \qquad |H| = \frac{\frac{f}{f_p}}{1 + \left(\frac{f}{f_p}\right)^2}$$

at band center $f = f_p$ peak value and phase zero

$$\begin{cases} |H(f_p)| = \dfrac{1}{2} \\ \arg H(f_p) = 0 \end{cases}$$

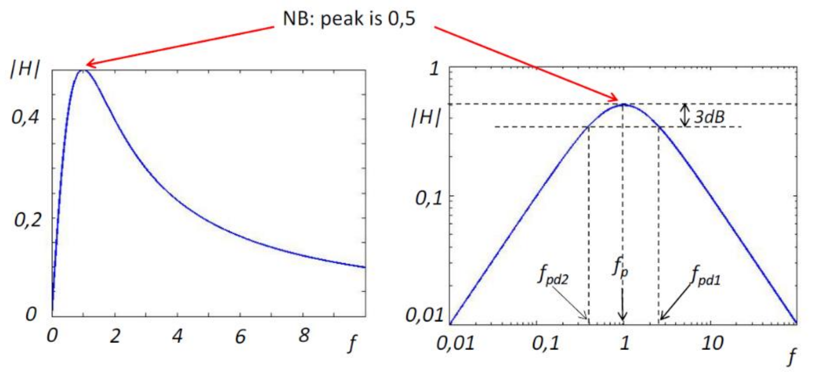
We are interested in the module of the delta response for the time domain and in the Fourier transform of the delta response (absolute value and squared for the noise). We will consider also the phase.
The resulting t.f. for the BP filter has the following key parameters we are interested in:

- Value at the central frequency.
- Phase in the central frequency.
- Bandwidth. We need to define it because it is difficult to define the BW if the shape of a signal or a filter is not a rect.

The peak value of the delta response at the central frequency is ½, but we would like to have 1, even if the gain is not important (high gain allows to neglect the noise of the following stage), while the phase is 0, so it is good, it seems that the combination of these two filters work well.

Let's plot the Fourier of the delta response.

Plots of $|H(f)|$ **with $f_p = 1$**



Linear – linear plot
|H(f)| vs. f

Log – Log plot (Bode plot)
|H(f)| vs. f

To understand the BW of the filter it is better to use the Bode diagram. In general, for the LP and HP filter, the BW is defined as the frequency at which the gain drops of 3 dB. We can make the same thing also for the BP filter (right plot).

BANDWIDTH

We have to take the modulus, define the peak value and go down of 3 dB with respect to ½.

$$|H| = \frac{\frac{f}{f_p}}{1+\left(\frac{f}{f_p}\right)^2} \qquad \text{peak value at } f=f_p \qquad |H(f_p)| = \frac{1}{2}$$

$$\text{3dB down points } f_{pd1} \text{ and } f_{pd2} \qquad |H(f_{pd1})| = |H(f_{pd2})| = \frac{|H(f_p)|}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$x = \frac{f}{f_p} \longrightarrow \frac{x}{1+x^2} = \frac{1}{2\sqrt{2}} \longrightarrow x_{1,2} = \sqrt{2} \pm 1$$

x $$f_{pd1,pd2} = (\sqrt{2} \pm 1) f_p$$

$$\text{3dB down pass-band} \qquad \boxed{\Delta f_p = f_{pd1} - f_{pd2} = 2 f_p}$$

$$\text{NOT narrow-band !!} \qquad \boxed{\frac{\Delta f_p}{f_p} = 2}$$

In the end we get the result x. The final result is that the BW is two times the central frequency, and this is not a narrow band, so we don't have a narrow band filter. This is a big problem from the signal point of view, but the purpose of the BP is to collect a small amount of 1/f noise, so we are interested more in the noise → let's study the same thing from the noise standpoint.

NOISE COMPUTATIONS

We have to study the absolute value squared of the Fourier transform of the delta response. Let's consider a unilateral spectral density and compute the ENBW, that allows me to design the noise in the frequency domain as a rect. |H(fp)| is the central value, delta(fn) the ENBW.

From the definition of white noise bandwidth Δf_n (**with unilateral S_B**)

$$\overline{n_B^2} = S_B \cdot |H(f_p)|^2 \cdot \Delta f_n = S_B \cdot \frac{1}{4} \cdot \Delta f_n$$

by comparison with the computed* output power

$$\overline{n_B^2} = S_B \cdot \int_0^\infty |H(f)|^2 \, df_n = S_B \cdot f_p \cdot \frac{\pi}{4}$$

we get $$\boxed{\Delta f_n = \pi f_p = \frac{1}{2T} = \frac{\pi}{2} \Delta f_p}$$

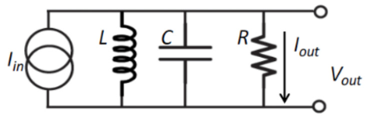
* Noise computation

$$\overline{n_B^2} = S_B \cdot \int_0^\infty \frac{\left(\frac{f}{f_p}\right)^2}{\left[1+\left(\frac{f}{f_p}\right)^2\right]^2} df = S_B f_p \cdot \int_0^\infty \frac{x^2}{[1+x^2]^2} dx = S_B f_p \cdot \int_0^\infty \frac{2x}{[1+x^2]^2} \cdot \frac{x}{2} dx$$

$$= S_B f_p \cdot \left\{ \left| -\frac{x}{2} \frac{1}{1+x^2} \right|_0^\infty + \frac{1}{2} \int_0^\infty \frac{1}{1+x^2} dx \right\} = S_B f_p \cdot \frac{1}{2} |arctg x|_0^\infty = S_B f_p \cdot \frac{\pi}{4}$$

We have to compute the value of the noise in the real case.
The result is that the central value is ok, the shift in phase is perfect (0), the BW from the signal point of view is not good, but also from the noise point of view (pi * central frequency). For example, for 1 kHz we have 300 kHz of noise BW compared to the 10 Hz used as an example in the previous video.

LCR RESONANT FILTER



$$Z = \frac{V_{out}}{I_{in}} = \frac{1}{j\omega C + \frac{1}{j\omega L} + \frac{1}{R}} = R\frac{j\omega \frac{L}{R}}{(1-\omega^2 LC) + j\omega \frac{L}{R}}$$

Denoting by $H(\omega) = \frac{I_{out}}{I_{in}} = \frac{1}{R}\frac{V_{out}}{I_{in}} = \frac{j\omega \frac{1}{RC}}{\left(\frac{1}{LC} - \omega^2\right) + j\omega \frac{1}{RC}}$ we have $\boxed{Z = R \cdot H(\omega)}$

At the **resonance frequency** $\boxed{\omega_o = \frac{1}{\sqrt{LC}}}$ the reactive impedances cancel each other

so that the impedance is purely resistive $Z(\omega_o) = R$

that is $H(\omega_o) = 1 \quad and \quad \arg H(\omega_o) = 0$

Another basic parameter is the **characteristic resistance R_o**, which for the oscillation at $\omega = \omega_o$ represents the ratio

(amplitude of voltage on C) / (amplitude of current in L) $\boxed{R_o = \sqrt{\frac{L}{C}}}$

It can be designed in a parallel way or in a series way. Z is the impedance of the structure; we want to study the t.f. of this filter, computed as the ratio between the current output and the current input. We are lucky because Vout/Iin is the impedance Z, that we can plug in the formula.

Parameters

One is the resonance frequency, which is the frequency that allows to have a purely reactive impedance for Z. Moreover, the absolute value of the t.f. at the resonant frequency is 1, and the phase is 0. The other is the characteristic resistance Ro, which is the ratio between the amplitude of the voltage on the capacitor and the current in the inductor.

We can also define alpha_0.

Starting from the poles: $\boxed{s_{1,2} = -\alpha_o \pm \sqrt{\alpha_o^2 - \omega_o^2}}$ $\boxed{\omega_o = \frac{1}{\sqrt{LC}}}$ $\boxed{\alpha_o = \frac{1}{2RC}}$

We can study the behavior of its δ-response:

The δ-response h(t) is:

- damped (real poles) if $\omega_o^2 < \alpha_o^2$, that is $R^2 < \left(\frac{R_o}{2}\right)^2$
- critically damped (coincident real poles) if $\omega_o^2 = \alpha_o^2$, that is $R^2 = \left(\frac{R_o}{2}\right)^2$
- oscillatory (complex poles) if $\omega_o^2 > \alpha_o^2$, that is $R^2 > \left(\frac{R_o}{2}\right)^2$

The **higher is R** with respect to R_o, the **lower is the dissipation**

and the **slower the damping** of the oscillation

If we study the behaviour of the delta response of this function in the time domain, we will find that it has 3 different behaviours, since we have 3 different parameters. So the delta response can be:

- Damped, with real poles.
- Critically damped, two coincident poles.
- Oscillatory, two c.c. poles.

So the **higher the value of the resistance, the lower the dissipation of energy** and **the slower the damping of the oscillations**.
The dissipation and energy give us the possibility to define the resonant quality factor Q.

Resonant quality factor – Q

The energy *E* stored in the circuit oscillates from C to L and back while it decays exponentially due to dissipation in R.

The lower is the loss rate, the higher is the resonator quality.
The reciprocal of this loss rate is defined **Quality Factor Q** of the resonator

that is $\boxed{-\frac{1}{E}\frac{dE}{d\vartheta} = \frac{1}{Q}}$ we can calculate $\boxed{Q = \frac{\omega_o}{2\alpha_o} = \frac{R}{R_o}}$

The **higher** is R >> R_o the lower is the dissipation ($Q \rightarrow \infty$ for $R \rightarrow \infty$)

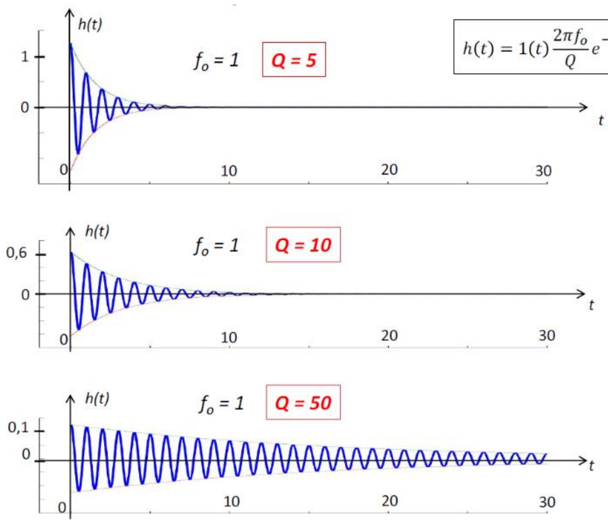
the transfer can be expressed in terms of resonance frequency ω_o and quality factor Q

$$\boxed{H(\omega) = \frac{j\omega \cdot \frac{\omega_o}{Q}}{(\omega_o^2 - \omega^2) + j\omega \cdot \frac{\omega_o}{Q}}}$$

We can more simply define Q = R/Ro. So the higher R compared to Ro, the lower the dissipation, because Q goes to infinite. We can also express the t.f. as a function of Q.

The responses as a function of the Q are the following. If Q is small, the response is damped.

$$\boxed{h(t) = 1(t)\frac{2\pi f_o}{Q}e^{-\frac{\pi f_o t}{Q}}\cos\omega_o t}$$



Increasing Q we are increasing the tau of the envelope in which the oscillations are confined and damped. For Q = inf the oscillations will be self-sustained.

Phase

The following is the phase of the delta response in the frequency domain. For omega that goes to +- inf the absolute value goes to 0, and this is good for the BP, while the phase goes to -+90°.
The phase at the central frequency is instead 0. We are also introducing another parameter, that is how much the phase changes as a function of omega as a function of the quality factor.
The variation of the phase with respect to omega is proportional to Q.

$$\boxed{\varphi = \arg H(\omega) = arctg\left[\frac{Q}{\omega\omega_o}(\omega_o+\omega)(\omega_o-\omega)\right]}$$

For $\omega \rightarrow +\infty$ $\quad |H| \rightarrow 0 \quad \varphi = \arg H(\omega_o) \rightarrow -\pi/2$ (- 90°)

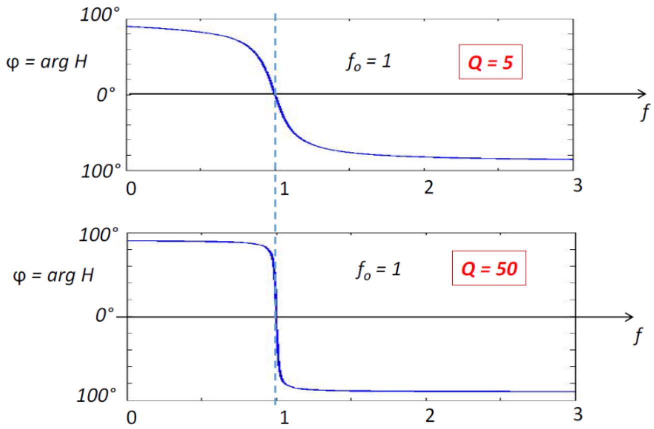
For $\omega \rightarrow -\infty$ $\quad |H| \rightarrow 0 \quad \varphi = \arg H(\omega_o) \rightarrow +\pi/2$ (90°)

For $\omega = \omega_o$ $\quad H(\omega_o) = 1 \quad \varphi = \arg H(\omega_o) = 0$ and $\boxed{\left(\frac{d\varphi}{d\omega}\right)_{\omega=\omega_o} = -2\frac{Q}{\omega_o}}$

The phase impressed by the filter is exactly zero at exactly the band center, but rapidly increases as ω is shifted.

Note that the higher is Q the steeper is the increase $\left(\frac{d\varphi}{d\omega}\right) \propto Q$

Let's plot the phase.



If Q increases, the phase is much steeper.
In fact, the idea of the BP is that we start from a signal that is modulated at a certain frequency and so we would like to put the BP exactly at the modulation frequency. But if the modulation frequency is not constant, if we have a steep behaviour, as soon as we move a bit the frequency we change a lot the phase, and this can be a big issue → the higher the Q the smaller the BW compared to the central frequency (good) but as a drawback, the phase becomes much steeper.

Module

We want to study what happens at LF, HF and around the central frequency.

$$|H(\omega)|^2 = \frac{\omega^2 \cdot \frac{\omega_o^2}{Q^2}}{(\omega_o^2 - \omega^2)^2 + \omega^2 \cdot \frac{\omega_o^2}{Q^2}} = \frac{1}{1 + Q^2 \left(\frac{\omega - \omega_o}{\omega_o}\right)^2 \left(\frac{\omega + \omega_o}{\omega}\right)^2}$$

- **«Lower wing»** approximation valid for $\omega << \omega_o$

  $|H(\omega)| \approx |H_L(\omega)| = \frac{\omega}{\omega_o}\frac{1}{Q}$ i.e. $\boxed{|H_L(\omega)| \propto \omega}$

- **«Higher wing»** approximation valid for $\omega >> \omega_o$

  $|H(\omega)| \approx |H_H(\omega)| = \frac{\omega_o}{\omega}\frac{1}{Q}$ i.e. $\boxed{|H_H(\omega)| \propto \frac{1}{\omega}}$

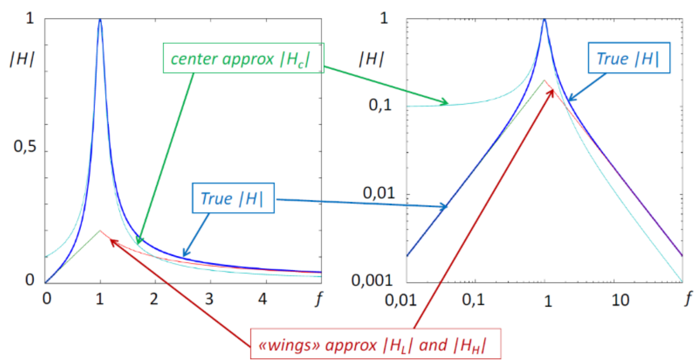
- **«Central lobe»** approximation valid for $|\omega - \omega_o| << \omega_o$, that is for $\omega \approx \omega_o$

$$\boxed{|H(\omega)|^2 \approx |H_C(\omega)|^2 = \frac{1}{1 + 4Q^2 \left(\frac{\omega - \omega_o}{\omega_o}\right)^2}}$$

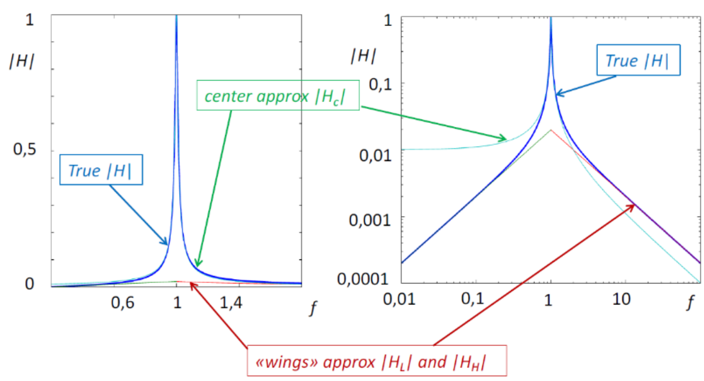
We are interested in the BW of the filter, that we can suppose is something around the central frequency, so we can study the central lobe.

Let's take some examples for the LRC t.f..

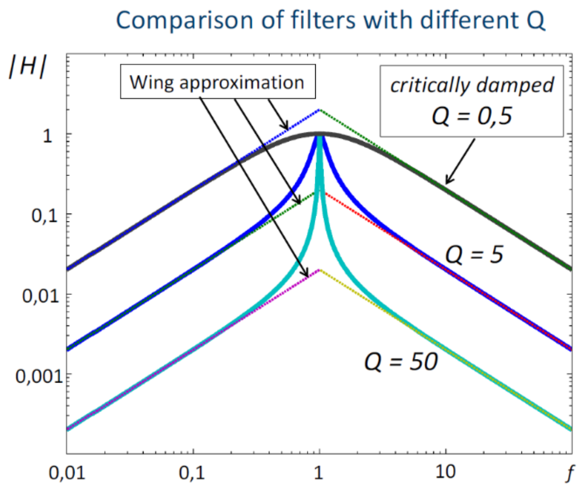
$f_o = 1$ **Q = 5**



$f_o = 1$ **Q = 50**



Increasing Q we can notice that we have a smaller BW for the filter, which is what we would like to obtain.



Signal Bandpass

Defined within the points where we have 3 dB down compared to the central value.

Bandwidth for signals: defined by the 3dB down points ω_{dL} and ω_{dH} where $|H(\omega_{dL})|^2 = |H(\omega_{dH})|^2 = ½$

$$\Delta\omega_s = \omega_{dH} - \omega_{dL}$$

For cases with Q >> 1 we can use the central lobe approximation

$$|H_C(\omega_d)|^2 = \frac{1}{1 + 4Q^2\left(\frac{\omega_d - \omega_o}{\omega_o}\right)^2} = \frac{1}{2}$$

and we find $\omega_{dH} - \omega_o = \omega_o - \omega_{dL} = \frac{\omega_o}{2Q}$

The signal bandwidth thus is $\boxed{\Delta\omega_s = \frac{\omega_o}{Q}}$ $\boxed{\frac{\Delta\omega_s}{\omega_o} = \frac{\Delta f_s}{f_o} = \frac{1}{Q}}$

Two basic advantages with respect to the CR-RC bandpass filter are quite evident:

- **No** signal attenuation at the center frequency
- **Narrow** filtering bandwidth even with moderately high Q values

For Q >> 1 we can use the central lobe approximation to solve the equation. This because we are interested in a very small BW.

So the basic advantages of the resonant filter compared to the CR/RC BP filter are that:

- No signal attenuation at the center frequency, where the modulus value is 1.
- Narrow filtering BW even with moderate Q values.

Noise Bandpass

We want to write the noise as Sb*delta(f), so as a rect. In this case the value in 0 is 1, so we have just Sb*delta(f) and not Sb * value in central frequency * delta(f).

The bandwidth for white noise is defined by

$$\Delta f_n = \int_0^{\infty} |H(f)|^2 \, df$$

In cases with Q >> 1 we can use for H(f) the central lobe approximation and take into account that $|H_c(f)|^2$ is with good approximation symmetrical with respect to the band center f_0 , thus obtaining

$$\int_0^{\infty} |H(f)|^2 \, df \approx 2\int_{f_o}^{\infty} |H_C(f)|^2$$

and therefore

$$\Delta f_n = 2\int_{f_o}^{\infty} \frac{1}{1+4Q^2\left(\frac{f-f_o}{f_o}\right)^2} \, df = \frac{f_o}{Q}\int_0^{\infty} \frac{1}{1+x^2} \, dx = \frac{\pi}{2}\frac{f_o}{Q}$$

$$\boxed{\Delta f_n = \frac{\pi}{2}\frac{f_o}{Q} = \frac{\pi}{2}\Delta f_s}$$

PROS AND CONS

- **Real capacitors and inductors are not pure C and L**. Their equivalent circuits include also finite resistances that model the internal sources of energy dissipation that inherently limit the Q of resonant circuits.
- In general, the dissipation is higher in components with higher value of L or C. **Good quality capacitors with low dissipation are available from pF to about 1 µF**. Inductors are more problematic than capacitors. **Good quality components are available from nH to a few 100nH**. Even components with fairly small L (typically a few 10 nH) have non negligible internal resistance.
- **Stray reactances must not be overlooked**. In discrete circuitry stray capacitances are in the order of pF and stray inductances are in the order of nH. In integrated circuits the values are much smaller, thanks to the very small physical size of the components.
- Since the resonance is at $f_o = \frac{1}{\sqrt{2\pi LC}}$, **for obtaining a low frequency f_o high values of both L and C are required**: in fact, with C=1 µF and L= 100 nH one gets f_o =1,26 MHz. Therefore, the Q values really obtained in the tuned filters progressively decrease as the desired resonant frequency decreases.

  - **For high frequencies f_o > 100MHz values of Q > 10 are currently obtained, up to almost Q ≈ 100** with clever design and high quality components.
  - **For intermediate frequencies 1MHz < f_o < 100MHz values up to Q ≈ 10** are obtained with careful design and implementation
  - **For f_o < 1MHz it becomes progressively more difficult to obtain high Q values** as the frequency decreases. Anyway, even with moderate Q the performance of the tuned filters is remarkable and in many practical cases filters with **Q ≈ 5** are really satisfactory.
  - For a given Q, note that the noise bandwidth is reduced as the resonant frequency f_o is reduced: $\Delta f_n = \frac{\pi}{2}\frac{f_o}{Q}$.

Constant-parameter tuned filters are a simple and economical solution, widely employed in prefiltering stages and other simple situations, but their use in high-performance filtering is hindered by some intrinsic drawbacks.

- **The accuracy and relative stability of f_o directly depends on that of the C and L values. Drift of f_o due to aging and temperature must be kept smaller** than the filter bandwidth, in order to avoid uncontrolled variation of the output signal amplitude and phase. This may be really difficult in case of very narrow bandwidth. In particular, strong phase variations are caused by even small variations of f_o because of the strong dφ/df at band-center of filters with high Q
- **Cascading simple filter stages for improving the cutoff characteristics is not practical** for narrow-band filters, because they should have very accurately equal and stable f_o.
- **The value of C influences both the center frequency f_o and the bandwidth Δf_S** , so that it is not easy to design a filter with specified f_o and specified Δf_S .
- **It is even more difficult to design a filter with adjustable f_o** and constant bandwidth Δf_S , as it is required for measuring power spectra and for other applications.
- In cases where the frequency of a narrow-band signal is not very stable, a filter with very narrow bandwidth can be employed only if its center frequency can be adjusted to track that of the signal. As above outlined, this is not easy to obtain.

ASYNCHRONOUS MEASUREMENT OF SINUSOIDAL SIGNALS

It is not possible a RLC filter with a high Q factor and it is not possible to create a good BP filter at low frequency (where LF is hundreds of kHz).

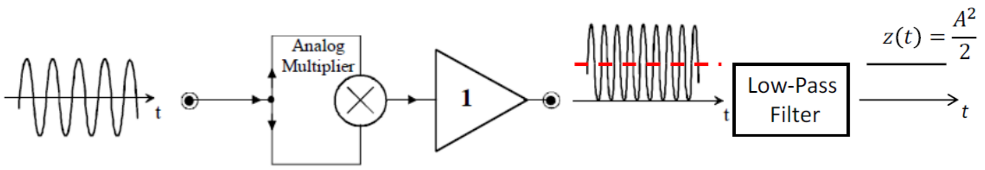
One idea could be to change the approach. We can resort back to the time domain and study the ways to detect the amplitude of a sinusoidal waveform in the time domain.
In the asynchronous approach (that we won't use) we have a cosinusoidal with a certain amplitude and in some cases we know the frequency, but not the amplitude. An example is the PS in our network. We have three different approaches, as below

- **Asynchronous** (or phase-insensitive) techniques were devised for measuring a sinusoidal signal without needing an auxiliary **reference** that points out the peaking time (i.e. the phase of the signal).
- They are currently employed in **AC voltmeters and amperometers**.
- The basic circuits of such meters are
  - the mean-square detector
  - the half-wave rectifier
  - the full-wave rectifier
- For a correct measurement of the amplitude of the sinusoidal signal, it is necessary to avoid feeding a DC component to the input of an asynchronous meter circuit. Therefore, the meter must be preceded by a filter that cuts off the low-frequencies, that is, a band-pass or a high-pass filter.

Mean-square detector

I want to measure the amplitude, I'm not interested in the phase. I take the square of the sinusoidal, and we remove the HF part (cosine) with a LP filter. This is good because I can get the amplitude.

$x(t) = A\cos(\omega t + \vartheta)$ $\quad\quad$ $y(t) = A^2\cos^2(\omega t + \vartheta) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega t + 2\vartheta)$



- It is a power-meter: the output is a measure of the **total** input mean power, sum of signal power (proportional to the square of amplitude A^2) plus noise power.
- The low-pass filter has NO EFFECT OF NOISE REDUCTION. In fact, it does not average the input, it averages **the square** of the input.
- For improving the S/N it is necessary to insert a filter **before** the Mean-Square Detector

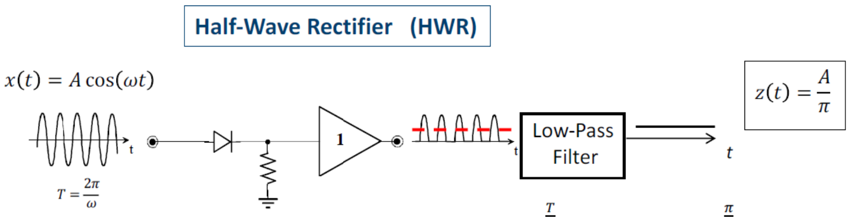
The problem is that the output is a measure of the total input mean power. This means that if we have an offset we are also measuring the amplitude of the offset, it is not filtered by the LP filter. Normally the offset is in the signal, if e.g. it is not modulated in a symmetrical way.

More importantly, we are using a LP filter to remove the frequencies at two times the frequency of the signal, not to increase the SNR, so also the noise is multiplied by itself and included in the constant value. So the SNR of this approach is not very high.

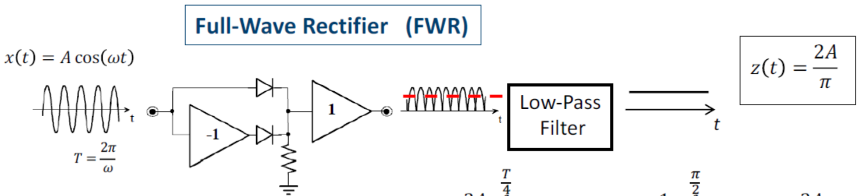
Hence one ideal is to find a new way to compute the amplitude of the cosinusoidal. In fact, if we find a way to compute it, we don't need the BP filter anymore.

Rectifier

The idea is to use a half-wave rectifier. So we rectify the waveform, we compute the average value with the LP filter and in the end the average of the rectified amplitude is A/pi. It is a constant output proportional to the amplitude.
We can also make a full-wave rectifier. The point is that the diode is like a comparator, that checks if the signal is positive or negative. It works perfectly theoretically, but we have also to consider its noise. Moreover, also the sum of the signal and the noise goes through the full-wave rectifier, so the crossing of the zero to activate the diode depends on the level of the noise.



$$z = \frac{A}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(\omega t)\, dt = A\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi = \frac{A}{\pi}$$



$$z = \frac{2A}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \cos(\omega t)\, dt = 2A\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\, d\varphi = \frac{2A}{\pi}$$

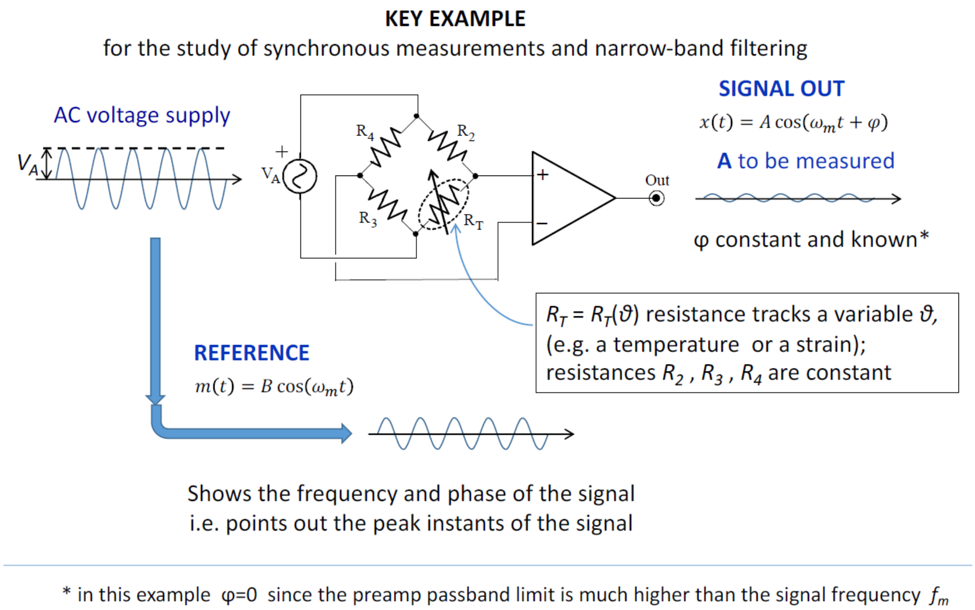
- The measurement with a rectifier is not really asynchronous, it is self-synchronized. The sinusoidal signal itself decides when it has to be passed with positive polarity and when passed with negative polarity (in the full-wave rectifier) or not passed at all (in the half-wave rectifier).
- In such operation, the LPF reduces the contribution of the wide-band noise, thus improving the output S/N. However, this is true only if the input signal is remarkably higher than the noise, i.e. if the input S/N is high.
- As the input signal is reduced the noise gains increasing influence on the switching time of the rectifier, which progressively loses synchronism with the signal and tends to be synchronized with the zero-crossings of the noise.
- The loss of synchronization progressively degrades the noise reduction by the LPF. With moderate S/N the improvement due to LPF is modest; with low S/N it is very weak. With S/N < 1 there is no improvement, there is not even a measure of the signal: the output is a measure of the noise mean absolute value.
- In conclusion, meters based on rectifiers can just improve an already good S/N. They can't help to improve a modest S/N and it is out of the question to use them when S/N <1. For improving S/N it is necessary to employ filters before the meter.

SYNCRINOUS MEASUREMENT OF SINUSOIDAL SIGNALS

EXAMPLE

Let's consider a Wheatstone bridge. The PS is a sinusoidal waveform. This configuration is an easy way to move the signal to high frequency, simply modulating the frequency of the PS.
The problem of modulating the signal is that we have 1/f at LF, and we want to move only the signal at HF, not also the 1/f noise. So if we modulate the output we are modulating both the signal and the noise, so the idea is to modulate it at sensor level.

The voltage we apply is sinusoidal (or cosinusoidal) and we get still a sinusoidal signal in output whose amplitude changes as a function of temperature (the information is in the amplitude and not in the frequency because we are modulating the signal).

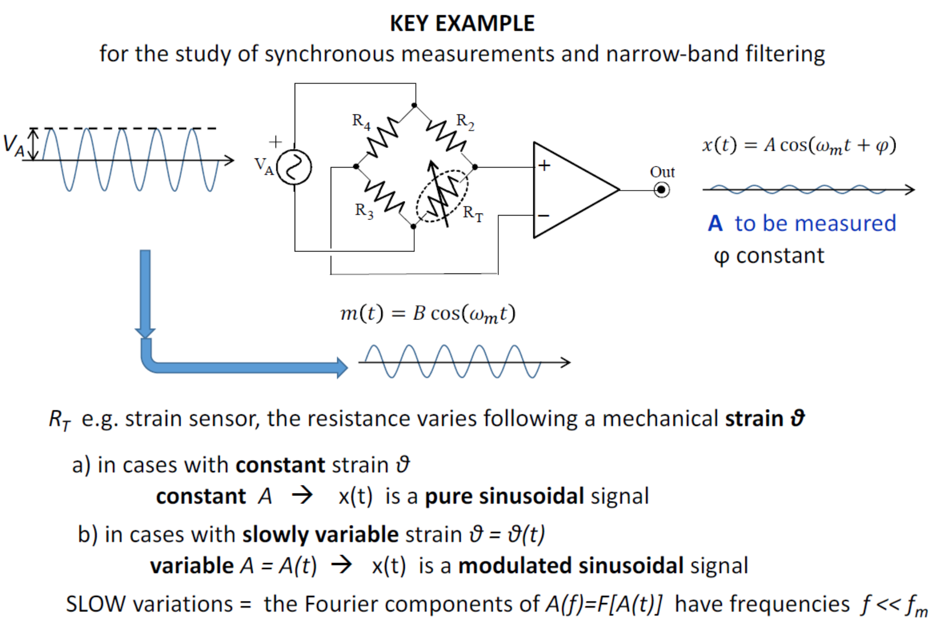


Since we are modulating the signal, we know the reference. How can we use this info to extract the amplitude of the sinusoidal?

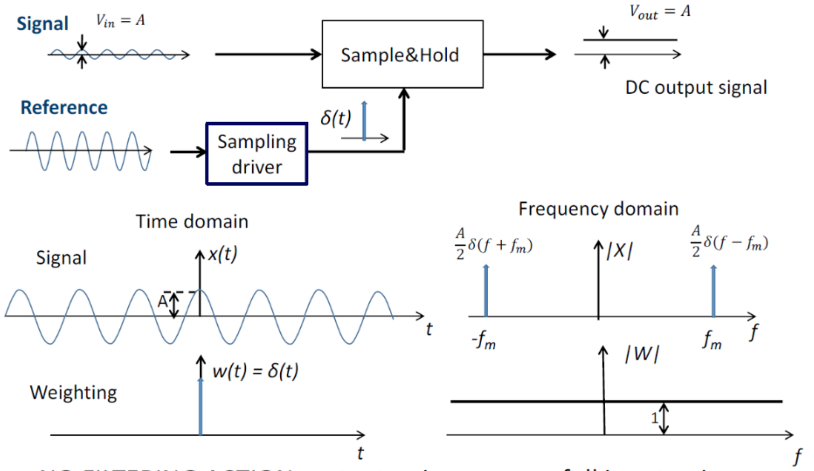
We want to move in the time domain, where we want to measure the amplitude of the sinusoidal.

A constant value of the resistive sensor gives us a sinusoidal in output with a constant amplitude. If it changes, the amplitude changes.

The goal is to extract a LF signal that is the amplitude of a signal at HF.



Peak sampling

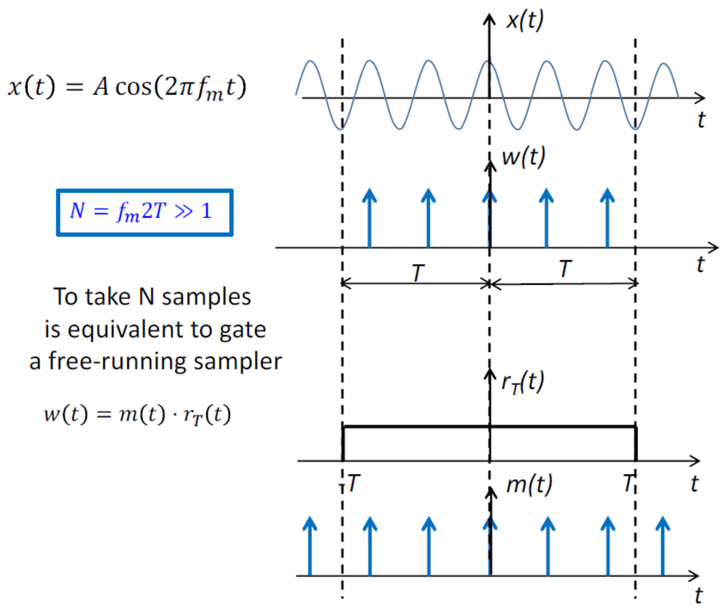


NO FILTERING ACTION: output noise power = full input noise power

We have to measure the amplitude and we have the sync, so we can use a S&H because we know where the peak is, so we sample the signal at the maximum amplitude. It is an approach that works.

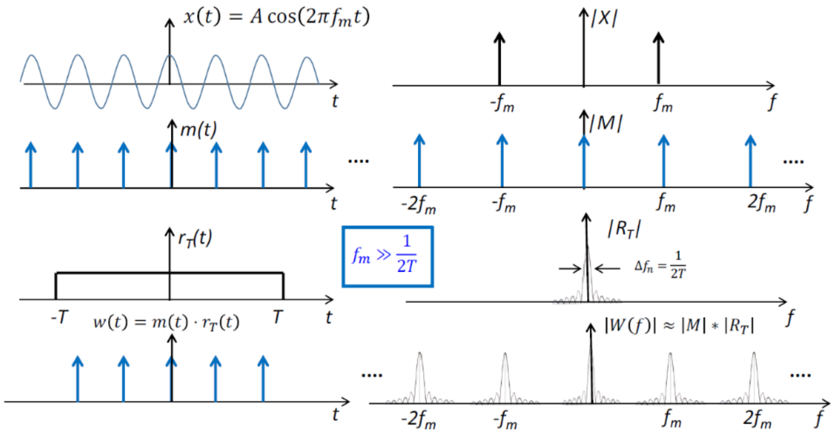
Let's consider an ideal delta for the sampling. We sample the signal and get the amplitude.
The problem is that of course we are getting the signal we want, but since we are using a delta we are also sampling noise, e.g. if we have a flat spectral density as for the WN, we are collecting a lot of it.

So we need to improve this solution to acquire less noise.
One possibility is to acquire more than just one sample as done in the case above, also because the amplitude is changing at LF. For instance, let's acquire five times the signal.



To create the w_f made out of delta in the time domain. Which is the Fourier transform of a w_f made of 5 deltas? It is difficult to say, so we derive it.
We take a delta, whose Fourier is known, and a comb of delta, whose Fourier is still a comb of delta. So we multiply the rect and the combo of delta in the time domain, hence convolution in the frequency domain.

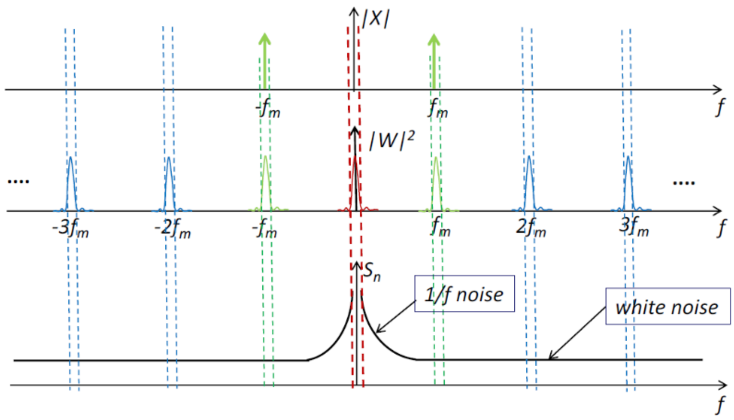
We want the frequency domain because noise detection in the time domain is though, it would require the autocorrelation computation, so it is much more easier in the frequency domain.



FILTERING: narrow bands at frequencies 0, f_m, $2f_m$, $3f_m$ ....

So the upper right plot is the Fourier of the sinusoidal, then we have the comb of delta. The formula in the blue box tells us that the distance between deltas in the frequency domain must be bigger than one over two times half the amplitude of the rect. This because the convolution in the frequency domain is not easy as in the time domain, since we need to consider the phase, but we can neglect the phase if one signal is composed just by lines and the width of the other signal is much bigger than the distance between two lines.

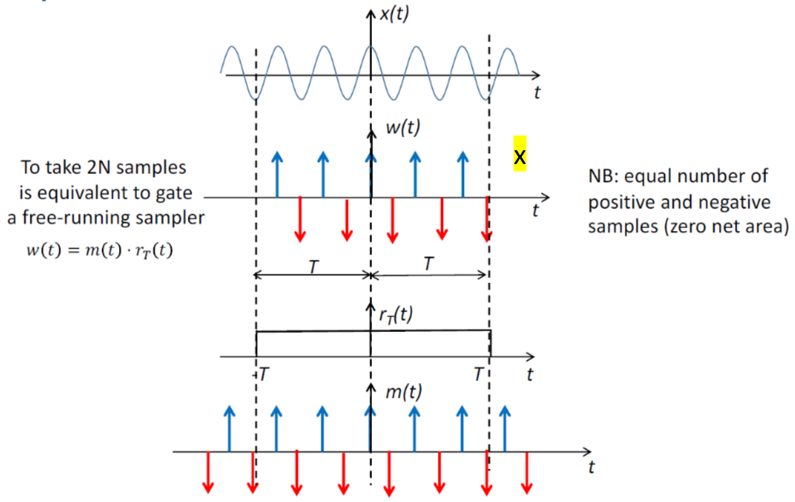
From the noise standpoint, we have WN and 1/f, and the 1/f can be neglected if we are after the frequency corner.
The green parts are where we are acquiring the signal, and noise is acquired instead in the green, blue and red regions. Compared with the 'flat line' of the previous case it is better, so we are acquiring less noise, but we have a problem; we are acquiring the signal in the green and the noise in all the other colors (just noise). But the real problem is not the blue, but the red.
In fact, we want to get rid of 1/f, but due to the red we are also collecting it, even if we are modulating to get rid of it. So the problem is not the blue BW that gives us more white noise, but the 1/f.



- At f_m **useful** band: it collects the **signal** and some white noise around it
- at f = 0 **VERY HARMFUL** band: it collects **1/f noise and no signal**
- at $2f_m$, $3f_m$, ... **harmful** bands: they collect just white noise without any signal

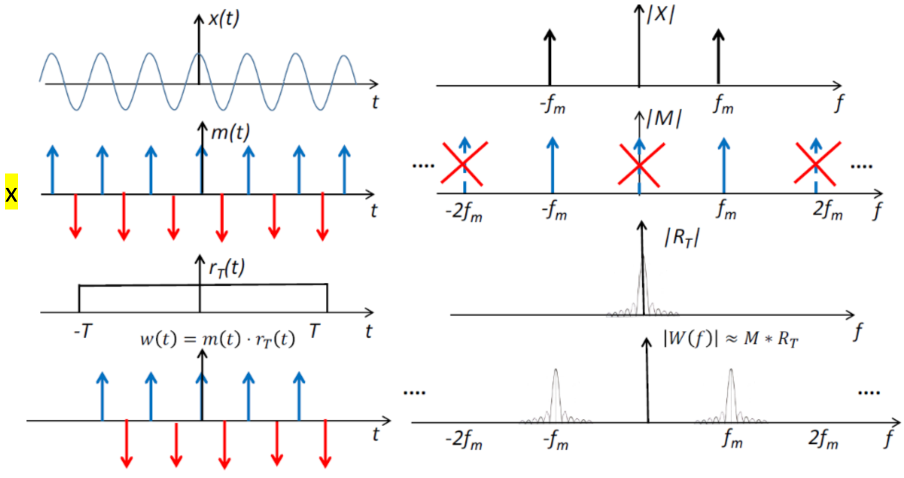
To solve this issue, we need to resort to signal theory.

DC suppression by summing positive peak and subtracting negative peak samples



We can try to use an approach similar to the correlated double sampling. We can create something like x, with positive deltas and negative ones. So we acquire 5 positive and 5 negative deltas. Differently from CDS that acquires baseline and signal, here we acquire 10 times the signal. As an advantage, we are not acquiring the zero frequency, because if we have a constant value (0Hz), 5 positive minus 5 negative gives 0 in output.

Now the comb of delta is made out of positive and negative deltas. How can we create it? As below.



FILTERING : narrow bands at f_m and at **odd** multiples $3f_m$, $5f_m$ ....

Which is the Fourier t.f. of positive and negative delta sampler?

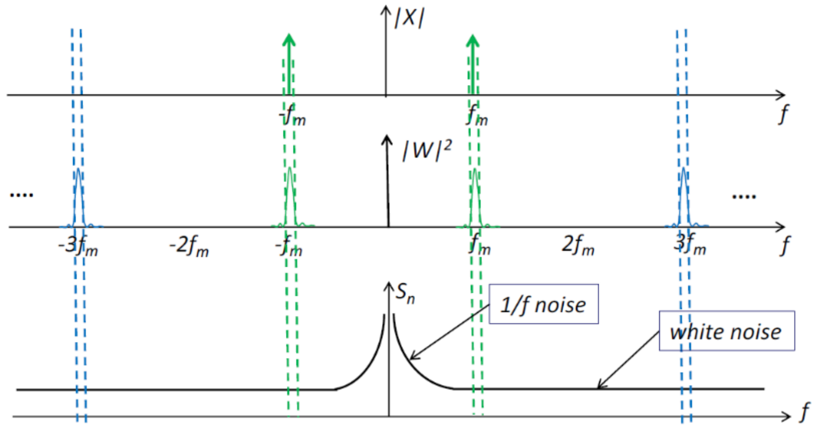
We know the Fourier of the blue free sampler, then I do 1 – it, and shift in time. The nightmare is that shifting in time deals with phase, which is a problem with a pen and paper approach (it would not be with MATLAB).

We could take the one peak of the blue and one of the read as a single CDS and replicate it in time domain, the problem is that still we are dealing with phase (replicating in time is sampling in the frequency domain).

Following the idea that making the replica in the time domain is sampling in the frequency domain and vice versa, we can say that a delta is a sample, so infinite positive and negative deltas are an infinite number of samples, because the distance between the different delta is constant. So we are sampling something a lot of time. Since the samples are constant, the signal we are sampling a cosinusoidal at two times the frequency.

The Fourier t.f. of the sampling in the time domain is the replica in the frequency domain. The Fourier of the cosinusoidal are two deltas in the frequency domain; now we are sampling in the time domain at two times the period, so we are replicating in the frequency domain. So we have fm and -fm, replicating them. The difference is that 0fm, 2fm, 4fm and so on are missing, because we are ‘replicating’ at odd multiples. So x is the sampling of a sinusoidal. Then the Fourier of the rect is the sinc and we move the sinc on all the different replicas.

Sample averaging with DC suppression



- at f_m **useful** band that collects the **signal** and some white noise around it
- **No more band at f = 0** , no more collection of 1/f noise
- at $3f_m$, $5f_m$ .... residual **harmful** bands that collect just white noise without any signal: how can we get rid also of them?

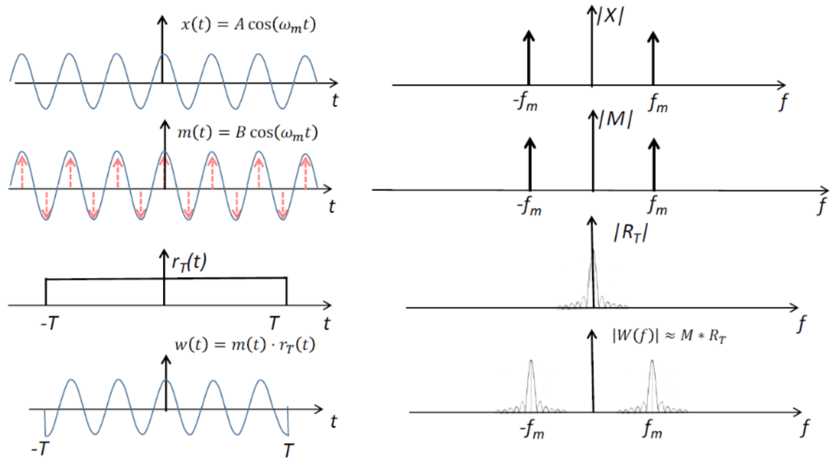
Every time we modify something we have to check that the original behavoiur of the filter is still there, so in this case that we are still acquiring the signal. In our case it is ok (green).
We are acquiring also the white noise (green and blue) but not at 0Hz, so no 1/f. The problem is that we are acquiring noise also at odd frequencies. Hence are we acquiring an infinite noise?
No, because at a certain point we will have a cutoff of the WN due to the LP filter or the preamplifier.

However, if we consider the replicas, theoretically the amplitude is doubled because we have two deltas overlapped. But we are happy because we are acquired both the signal and the noise two times. Then, when we normalize for the SNR, it is improved.

Continuous sinusoidal weighting

Further improvement. We want to match the w_f with the shape of the signal, hence creating a matched filter. We can try to use the theory of the optimum filter, so we use exactly a piece of the signal to get the deltas.
In the end I get the w_f x, which is a part of a cosinusoidal (signal truncated by the multiplication with the rect). It is truncated because we want to check the signal for some time, not for an infinite time, in case the amplitude is changing.

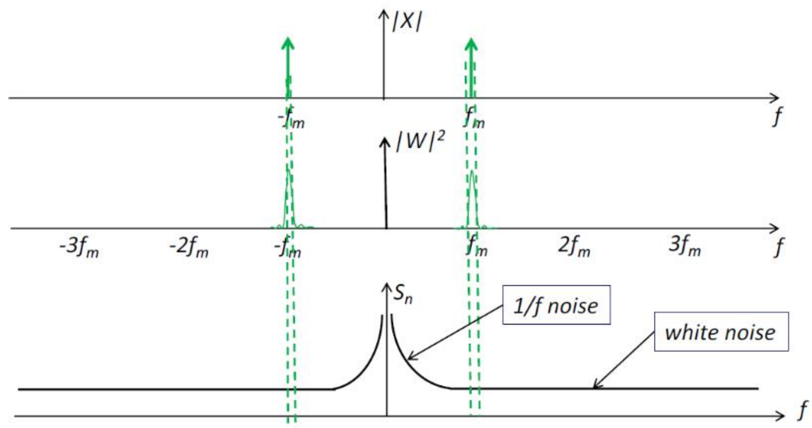
Continuous sinusoidal weighting instead of peak sampling



TRULY EFFICIENT FILTERING : just one narrow band at f_m

The final result is that we acquire the signal in the green BW, and also the noise, but we don't have the zero frequency noise and any other noise at other frequencies. So it seems we are acquiring a small BW around fm, which is exactly our goal, a narrow BP filter. It seems that this solution solves all the problems related to the BP filter.

Optimized noise filtering in synchronous measurement



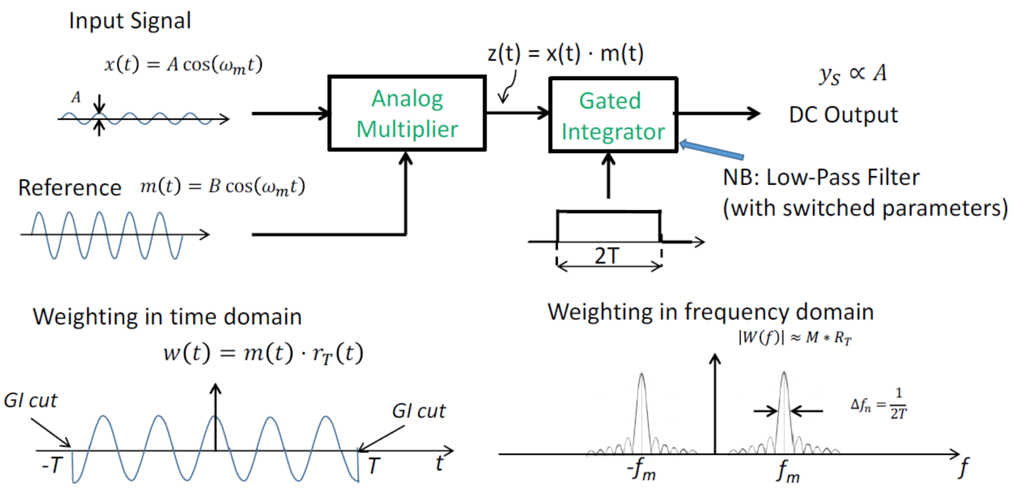
- at f_m **useful** band that collects the **signal** and some white noise around it
- No band at f = 0 , no collection of **1/f noise**
- No residual bands at $3f_m$, $5f_m$ ... no more collection of white noise without any signal

How to implement this optimized synchronous measurement?

Which is the BW of this created BP filter?

It is 2/T, where T is the observation time (width of the rect), so we are taking the first lobe of the sinc. So the BW is related to the width of the rect, while fm is related to the signal, which is uncorrelated with T. So we can change one or the other without having Q and BW of the BP filter related. So it seems we are solving the issues of the BP filter, at least in the frequency domain. But how can we implement this filter.

Filter implementation – synchronous measurement



- **NB** the reference input to the multiplier is a **STANDARD** waveform, which absolutely does **NOT** depend on the signal: it is the same for any signal !!
- Therefore the set-up is a **LINEAR** filter (with time-variant parameters)

We take the modulated signal, the reference, and we multiply the signal and the reference. After that, we apply a GI and we get the output.
Firstly, we need to check that we can do that. We need an analog multiplier to multiply the input signal and the reference. Also the GI can be implemented at the output, because we have the sync. The output will be the amplitude A.

Advantages

This linear time-variant filter composed by Analog Multiplier (Demodulator) and Gated Integrator (Low-pass filter with switched-parameter) has a weighting function similar to that of a tuned filter with constant-parameter, but has basic advantages over it:

- Center frequency f_m and width Δf_n are **independently** set
- The **center** frequency is set **by the reference** *m(t)* and locked at the frequency f_m
- In cases where f_m has not a very stable value the filter band-center tracks it: the signal is thus kept in the admission band even if the width Δf_n is very narrow.
- The **width** $\Delta f_n = 1/2T$ is set **by the GI**, it is the (bilateral) passband of the GI
- Narrow Δf_n and high quality factor Q can thus be easily obtained **at any f_m**

$$\Delta f_n \ll f_m \qquad\qquad Q = \frac{f_m}{\Delta f_n} \gg 1$$

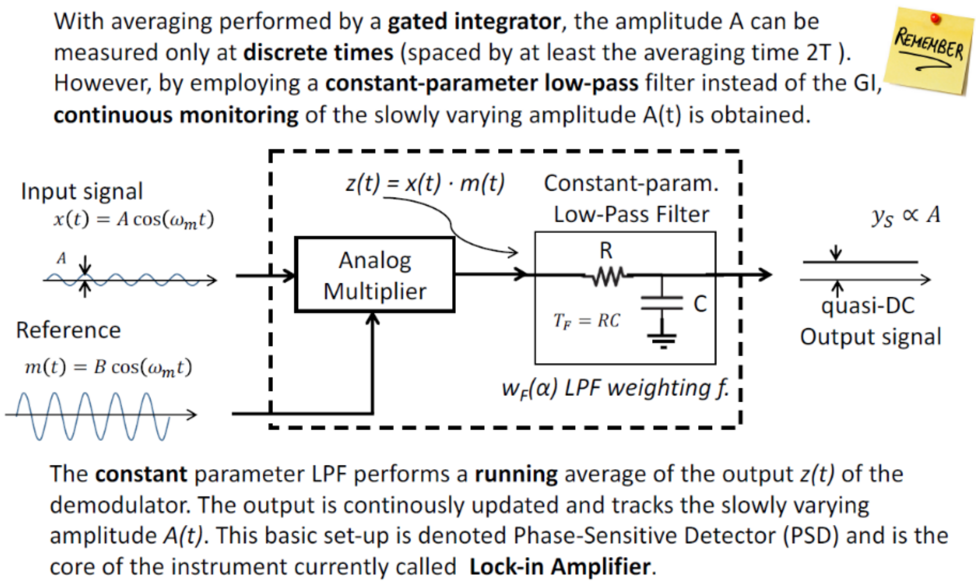
Now frequency and BW are totally independent, and the BW is just the width of a GI, so we can create any BW, and so a huge Q.

However, there is still a problem. E.g., if we are interested in measuring the temperature, which is a slowly changing function of time, with this filter we are multiplying the signal (which is modulated) with the reference and then the GI because we have the sync. The GI is a NCPF, and the output of any NCPF is a number. But I don't want the temperature 'now' but its evolution in time, because I started from a signal varying in time. So I would like a signal changing in time also at the output of the filter, otherwise I have to sample every time the system.

How can we get the entire waveform? Lock-in amplifier.

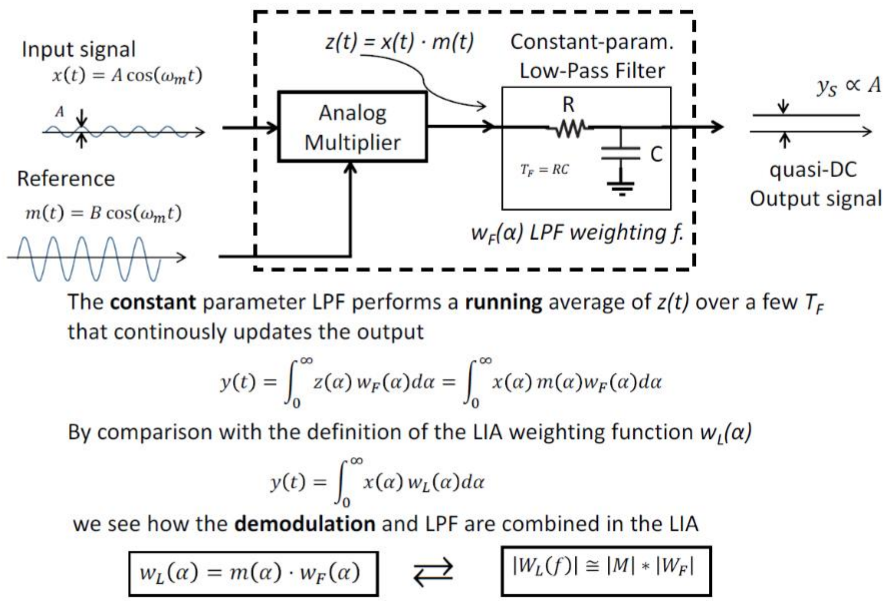
LOCK-IN AMPLIFIER

The idea is that still we take the signal and reference and we multiply them, but instead of using GI (NCPF), we replace it with a RC, which is a CPF. We started with a GI because we needed a piece of the sinusoidal function, and the GI has a rect w_f. Now we are modifying it hoping to have a continuous output. but does it work?
We need to check if this solution is doing the same thing of the previous one.



Weighting function

Z(t) is signal times the reference, and then we have a LP filter, so the output is z(t) times the w_f of the LP filter.

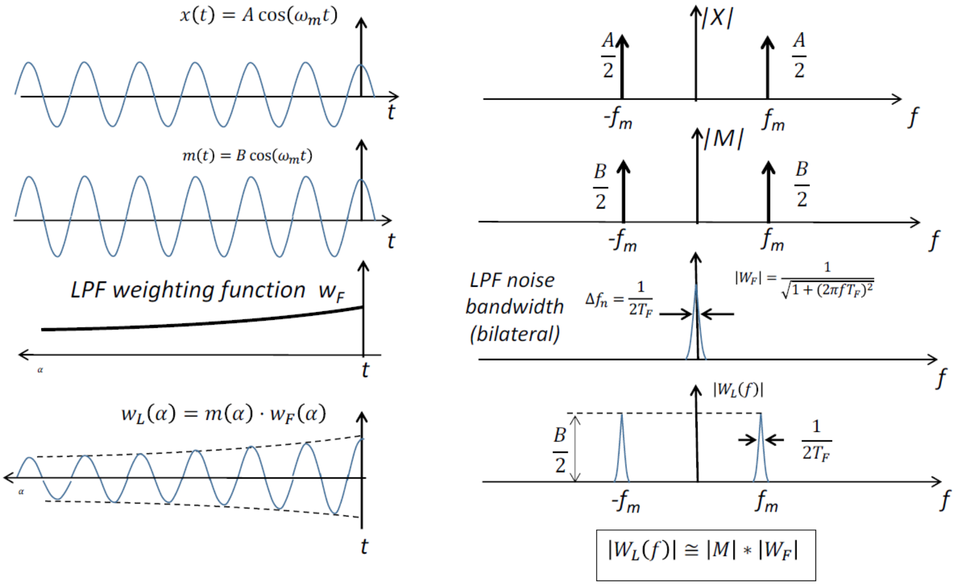


The output of any filter (CPF or NCPF) is the integral of signal times w_f, in the time domain. If we compare the two equations, we get that the w_f of the lock-in is the multiplication of reference times w_f of the LP filter.

Let's move to the frequency domain.

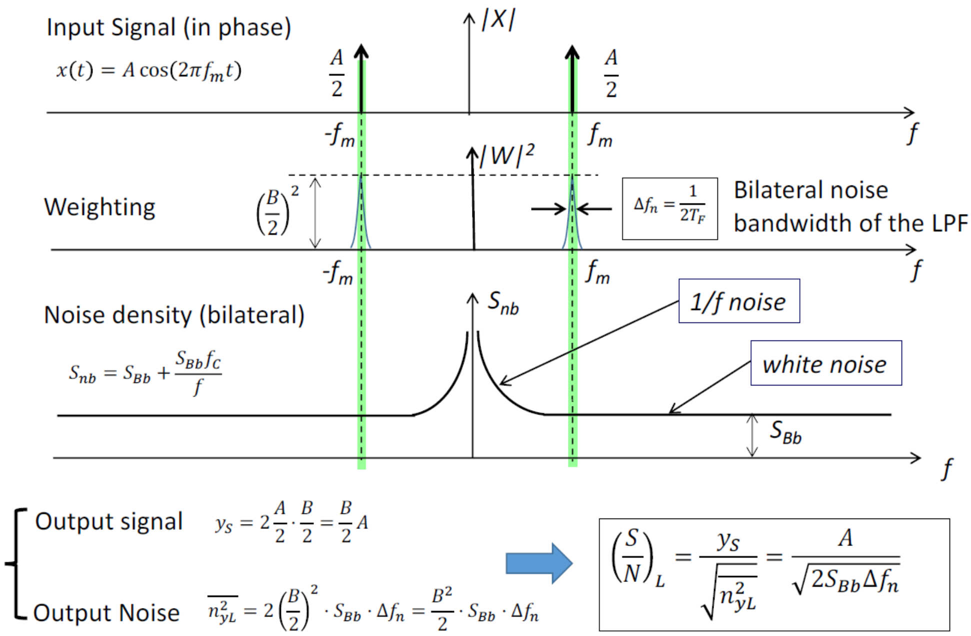
Frequency domain

In the time domain we are multiplying the reference and the w_f of the LP filter, so the last plot on the left is the w_f in the time domain.



The w_f of the LP filter in the frequency domain is the Lorentzian spectrum. So the situation is as before with the GI, but instead of a sinc we have the Lorentzian spectrum. The BW is not related to the width of the rect (we don't have it anymore) but to the RC of the LP filter, and the central frequency is still fm. So we have the same advantages as before and no disadvantage, with the difference that the output now is a function, and not just a number.

SNR of the lock-in amplifier



Signal is signal times the w_f, so 2*A/2*B/2 (w_f, not w_f^2 for the noise).
For the noise we need to take the absolute value squared for the w_f. In the image we have the bilateral spectral density.

$$\boxed{\left(\frac{S}{N}\right)_L = \frac{A}{\sqrt{2S_{Bb}\Delta f_n}}}$$

or in power terms $$\boxed{\left(\frac{S}{N}\right)_L^2 = \frac{\frac{A^2}{2}}{S_{Bb}\Delta f_n}} = \frac{in - phase\ signal\ power}{halfpower\ of white\ noise\ in\ the\ band\ \Delta f_n}$$

S/N equation in terms of the unilateral parameters

By introducing

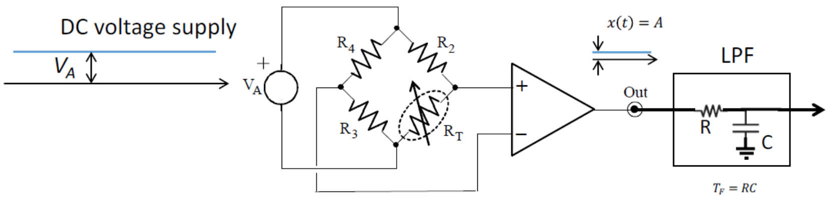
- f_{Fn} the LPF unilateral bandwidth (upper band-limit for noise), i.e. $\Delta f_n = 2f_{Fn}$
- S_{bu} the unilateral noise density, i.e. $2S_{Bb} = S_{Bu}$

we can write

$$\boxed{\left(\frac{S}{N}\right)_L = \frac{A}{\sqrt{2S_{Bu}f_{Fn}}}}$$

DC signal with LPF compared to AC signal with LIA

Let's resort back to the original problem, where we wanted to measure the temperature and we had the 1/f. Let's make a comparison applying everything as the beginning but with a constant value for the bridge, not sinusoidally modulated.



Let us consider the set-up of the key example (measurement with resistive sensor) now with DC supply voltage V_A equal to the amplitude of the previous AC supply. The signal now is a DC voltage equal to the amplitude A of the previous AC signal.

With a LPF equal to that employed in the previous LIA we obtain:

$$\begin{cases} \text{Output signal} \quad y_C = A \\ \text{Output Noise} \quad \overline{n_{yC}^2} = \widehat{S_{nu}} \cdot f_{Fn} \\ (\widehat{S_{nu}} \text{ mean density in the LPF band}) \end{cases} \Rightarrow \boxed{\left(\frac{S}{N}\right)_C = \frac{y_C}{\sqrt{\overline{n_{yC}^2}}} = \frac{A}{\sqrt{\widehat{S_{nu}}f_{Fn}}}}$$

This S/N may look **better by the factor $\sqrt{2}$** than the S/N obtained with the LIA, but is this conclusion true?

NO, such a conclusion is **grossly wrong because $\widehat{S_{nu}} \gg S_{Bu}$!!**

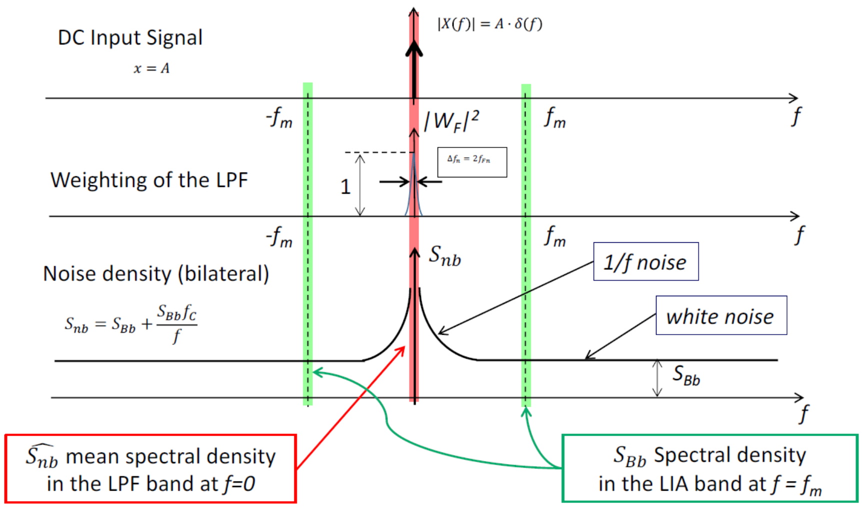
The signal is A, constant or very slowly varying in frequency. The noise is spectral density times ENBW, and we know the ENBW of the LP filter. Now we consider just the spectral density as the mean density in the LP filter band. We don't have delta(f) because we start from 0, so delta(f) = f_Fn.

We can set the RC with the same tau of the LIA.

It seems that we have a higher SNR without the LIA, because with LIA it's sqrt(2)/2 (like if we were doubling the noise). So what are we missing?

We are not doubling the noise with the LIA and the LP filter solution is worse than the LIA because the spectral density Snu_hat is not the same, since the LP filter BW is in the middle of 1/f, while in the LIA the spectral density was including jus the WN, not the 1/f noise.

So with just the LP filter we have the component around 0 (red), collecting the 1/f noise exactly when it's higher. Just the LP filter is not enough because we are sampling the 1/f in zero, so we need to add a LPF, so all the reasoning related to eventually CDS or baseline restorer (all the problems already seen for the 1/f noise).



A passband at $f = 0$ is a risk: 1/f noise gives $\widehat{S_{nb}} \gg S_{Bb}$!!

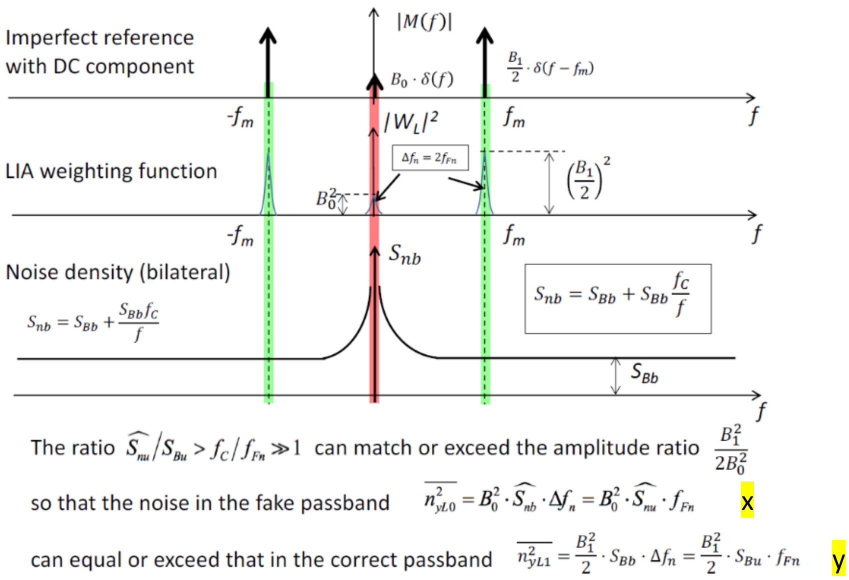
Fake LIA passbands arise from imperfect modulation

LIA implementation in the real world is not easy, especially when we have to create the analog multiplier. One of the problem with the analog amplifier are issues related to noise and dynamic ranges. Furthermore, it adds distortion to the signal, so 'adding lines'. If we are adding lines to the reference due to distortion we are adding BW where we are collecting noise. If this BW is in zero, we are collecting 1/f noise.

So normally we have a small DC value because we are not able to create a perfect reference.

- Ideally, the reference waveform should be a perfect sinusoid at frequency f_m with amplitude B_1
- In reality, deviations from the ideal can generate spurious harmonics at multiples kf_m ($k = 0, 1, 2 \ldots$) with amplitudes B_k (small $B_k \ll B_1$ in case of small deviations)
- Moreover, effects equivalent to an imperfect reference waveform can be caused by non-ideal operation (non-linearity) of the multiplier
- Since it is $$|W_L(f)| \cong M(f) * W_F(f)$$ each spurious harmonic component of $M(f)$ adds to the LIA weighting function W_L a spurious passband at frequency kf_m with amplitude B_k and shape given by the LPF
- A fake passband **at $f = 0$ is particularly detrimental even with small $B_0 \ll B_1$** because it covers the high spectral density of $1/f$ noise ....
- .... and unluckily any **deviation from perfect balance of positive and negative areas** of the reference produces a DC component with associated passband at $f = 0$!!

If we have a small line in 0, e.g. due to a small offset in the reference, we are opening a BW with small amplitude collecting some noise.
But we aren't necessarily damaging our work. We need to compare the noise x with the one of the signal, y. It depends on the value of B0.



From the theoretical point of view, the Lock-in is perfect because B0 doesn't exist. The problem is that a perfect cosinusoidal doesn't exist, as well as a perfect amplifier.

LOCK-IN AMPLIFIER

In principle with the LIA we can obtain a high SNR also when the SNR at the beginning is very small. The problems are the non-idealities of the real implementation of the filter.

In principle:

a LIA consisting simply in a **Phase-Sensitive Detector** provides a flexible and effective band-pass filtering that can achieve **very narrow bandwidth**. It is thus able to recover for measurement with good precision even very small modulated signals buried in much higher noise, down to an ideal limit value $S/N \ll 1$

But in practice:

the non-ideal features of the actual circuits of the PSD set to the recovery of small signals buried in high noise an actual limit much more stringent than the ideal one.

However:

by introducing in the LIA structure modifications and further stages, the hindering features can be counteracted and the actual detection limit can be improved towards the ideal limit. For instance, in real cases **nanoVolt signals** can be extracted from wideband noise with 1000 times (60dB) greater rms value.

Principles

The problem is not the gain of the filter, but of the multiplier.

High gain for the signal

The modulated input signal is converted by the LIA in a slow demodulated signal, with components from DC to a fairly low frequency limit. **This signal must be supplied to a meter circuit that measures its amplitude, i.e. nowadays ordinarily an ADC**. The LIA output signal must have scale adequate for the ADC (typically 10V full scale), whereas the LIA input signal is very small: therefore, the LIA **must provide high overall gain for the signal.**

Post-Amplifier (after the PSD)

A high-gain amplifier after the PSD (denoted here Post-Amplifier) is employed to raise the demodulated signal to a scale suitable for the ADC.
Notice that the post-amplifier:

1. must be a **DC-coupled** amplifier with upper bandlimit adequate to the demodulated signal
2. receives a signal accompanied by low noise, since it operates after the PSD filtering
3. It has drift of the baseline offset and low-frequency noise, which affect the measurement since they occur **after the PSD** and are **not filtered**

If the amplifier has a drift or offset, we pay all the 1/f of the amplifier. So using a post-amplifier just to increase the gain doesn't work. So maybe better to use a pre-amplifier.

Pre-Amplifier (before the PSD)

If the demodulated signal is very small, comparable or lower than the baseline drift and noise of the post-amplifier referred to its input, the measurement will be spoiled. A **preamplifier before the PSD** is necessary in order to avoid or reduce this drawback.

Notice that the pre-amplifier:

1. processes the modulated input signals, hence it is an **AC coupled** amplifier, either **wide-band** type including the modulation frequency f_m **or narrow-band tuned to** f_m
2. receives a signal accompanied by high noise, because it operates before the PSD
3. may have baseline drift and low-frequency noise, but their role is minor because they are filtered by the PSD (and by the AC-coupled amplifier itself).

WARNING: Signal and Noise MUST stay within the Linear Dynamic Range

In order to obtain the foreseen improvement of S/N, the processing of signal and noise in the LIA must be accurately linear. **Deviations from linearity produce detrimental effects** (self-modulation of the noise, generation of spurious harmonics, etc.), which irrevocably alter the measure and degrade the LIA performance. **The signal and noise must remain well within the linear dynamic range in every stage involved, particularly in the multiplier (and in the preamplifier).**

If the signal is larger, the output will also be larger, so I amplify at the input. I'm amplifying both the signal and the noise, so SNR is not touched, I'm just changing the amplitudes. The BW of the amplifier must be large (AC coupled).
Moreover, the preamp can have drift or 1/f, because it is then removed by the LIA. The only thing is that the preamp must be AC coupled, so it must include fm, frequency with which we modulate the signal.

Fm is typically higher than the frequency corner, which is normally in the range 1k – 10 kHz, so greater than 1 MHz. The problem is that signal and noise must stay between the linear dynamic range of the multiplier. If we exit it, we create distortion or clamp the signal.
With the preamp we are multiplying the signal and noise, signal is nV, but completely covered by noise, so if we move the signal to mV, we are moving the noise to V and the noise could be out of the dynamic range, while the signal is. As soon as we exit the dynamic range, the multiplier starts to distort everything.

So we cannot amplify at the output of the LIA nor at the input. So either we create an amplifier and multiplier with large dynamic range, or we change perspective.

The idea is to make something meaningless from the theory point of view. I use a preamp, which acts both on signal and noise, so the solution is to use a resonant filter RLC. We are not replacing the LIA with a resonant filter, we are adding it. The BW of the RLC is very large, while the one of the LIA is narrow, and the two are in series. But the idea is not to use the RLC to improve the SNR, but as a pre-filter.

We take the signal, we use a rough pre-filter with a large BW, but not so large as the input, so we cut some noise and then at this point the preamp will amplify the signal and the cut noise, and they will hopefully remain in the dynamic range of the analog multiplier.

Wide-Band Preamplifiers and Tuned Preamplifiers

When a **wide-band preamplifier** is employed to raise the level of a very small input signal, a problem arises with very small input S/N<<1. **The gain required for the signal works on a noise which is much higher than the signal, hence it brings this amplified noise out of the linear dynamic range of the multiplier.**

In such cases, for exploiting the required gain it is necessary to reduce the noise received by the preamplifier with a **pre-filter**. Adequate reduction of the LIA input noise is obtained in many cases with **prefilter passband much wider than that of the LIA**.

Such a prefiltering would be a useless nonsense in an ideal apparatus, but in real cases it is a necessary feature for avoiding nonlinearity in intermediate stages. On the other hand, we will see that a very narrow-band prefilter is not advisable.

Preamplifiers that incorporate prefiltering are currently available from LIA manufacturers; they are called **tuned preamplifiers or selective preamplifiers**.

Another possibility is to remove the analog multiplier.

Linearity limits and problems

- The **multiplier dynamic range** (linear behavior range) does not depend on the gain setting of the preamplifier, it is set just by the multiplier circuit.
- The **preamplifier output dynamic range** is constant, independent of gain setting.
- Therefore, there is a **maximum acceptable input signal** that must not be exceeded for maintaining linear behavior of preamplifier and multiplier; increasing the preamplifier gain by a factor decreases this limit by the same factor
- There is also a **maximum input rms noise** that can be applied maintaining linear behavior; an increase of preamp gain decreases also this limit.
- Also the **post-amplifier** has limited linear range, but problems met are much less severe. In fact, the post-amp receives low-level noise (filtered noise), whereas preamp and multiplier process high-level noise (not filtered or just prefiltered)
- Each setting of preamplifier and post-amplifier gains determines an **input full-scale signal**, i.e an input signal level that produces full-scale LIA output signal.
  Note, however, that a given value of input full-scale signal can be obtained with different combinations of preamp gain and post-amp gain

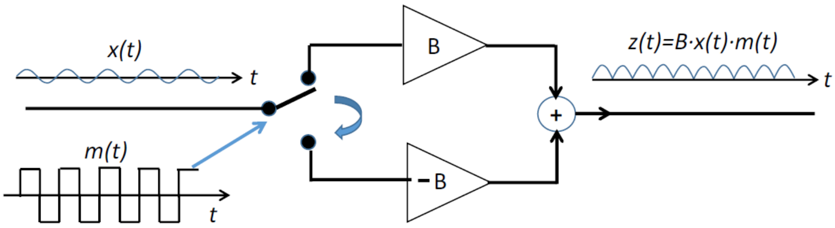
Elimination of the analog multiplier

The idea is that we have to multiply the signal and the reference, but it doesn't necessarily require an analog multiplier. If the reference is not sinusoidal, we can use two amplifiers and a switch.
At this point we don't have any problem about the dynamic range, because we removed the analog multiplier, but we have to be sure that the new solution works as the original LIA.

So we have to study the LIA, but with a square wave reference.

Switched Amplifier Circuits instead of Analog Multipliers

We have seen that modulation with squarewave reference *m(t)* can be implemented with circuits based simply on switches and amplifiers, avoiding recourse to analog multipliers



The noise referred to the input, the linearity and the dynamic range of these circuits are remarkably **better than those of analog multiplier circuits** (even high-performance types) because they are limited just by the performance of amplifiers and switch-devices.

Therefore, switched linear circuit configurations are often employed as demodulator stage in LIAs in order to avoid the limitations of analog multipliers.

The w_f of the LIA is the product of the w_f of the LP filter and the reference, but the reference doesn't have to be sinusoidal (convolution in the frequency domain still retrieves the considerations previously done on the phase). With one amplifier and we switch the inputs, we are sure that positive and negative values are the same, since the amplifier is the same. This is an advantage since we can kill the offset.

The weighting function $w_L(\alpha)$ of a LIA is the multiplication of reference waveform $m(\alpha)$ (periodic at frequency f_m) and weighting function $w_F(\alpha)$ of the LPF

$$\boxed{w_L(\alpha) = m(\alpha) \cdot w_F(\alpha)}$$

In frequency domain this corresponds to the convolution of the F-transforms

$$\boxed{W_L(f) = M(f) * W_F(f)}$$

Since:

a) the transform $M(f)$ of a periodic $m(\alpha)$ is composed by lines at f_m (fundamental) and integer multiple frequencies (harmonics)

b) $W_F(f)$ of the LPF has bandwidth much smaller than f_m

the result of the convolution of $W_F(f)$ by any line of $M(f)$ does not overlap the result by any other line (with very good approximation). We conclude that:

the $W_L(f)$ is a set of **replicas of $W_F(f)$** centered on each line of $M(f)$, **multiplied by the line-weight** and **phase-shifted by the line-phase**.

With very good approximation, the module diagram can thus be obtained simply as

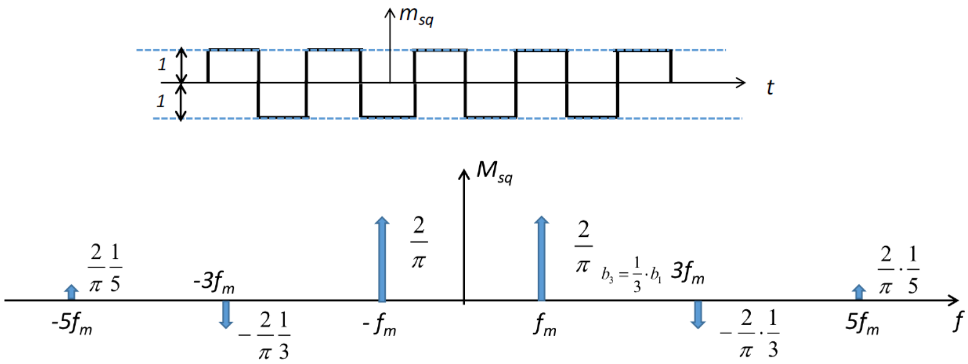
$$\boxed{|W_L(f)| \cong |M| * |W_F|}$$

The new reference is a square waveform.

Square waveform reference

Its Fourier t.f. is made by lines whose distance is much larger than the BW of the LP filter, so we can apply the simplified convolution disregarding the phase.

$m_{sq}(t)$ = symmetrical squarewave (from +1 to-1) at frequency f_m



- Only odd-harmonic sinusoidal components
- Components with alternately positive and negative sign, i.e. alternately 0 and π phase
- Component amplitude decreasing as the reciprocal order:
  $b_1 = \frac{2}{\pi}$ at fundamental f_m ; $b_3 = \frac{1}{3} \cdot b_1$ at $3 f_m$; $b_3 = \frac{1}{5} \cdot b_1$ at $5 f_m$; ......

Good things about a square wave reference:
- Reference composed by lines, so easy to make the convolution.
- It doesn't have the delta at zero frequency.

To create a small delta in 0Hz, we can create a mismatch between the positive and negative amplitude, or we can use a duty cycle that is not 50%. So, since these things hardly happen, this square waveform is a good candidate for being the reference.

The drawback is that we have more noise, replicas at odd fm multiples where we acquire the noise. So we are acquiring more noise than the one required. More than that, thanks to these replicas we will be able to get a SNR higher than the SNR obtained with a sinusoidal reference.

Recap

The problem is always the 1/f noise; CDS or CR are useful for the 1/f, but if 1/f is too high we have a problem. However, it is possible to modulate the signal outside the 1/f, meaning at a frequency higher than the frequency corner fc. Not at too high frequencies because we could have problems with the amplifier. If we have available a BP filter, we would be happy if the signal is not in the middle of 1/f as a central frequency.

The 1/f is infinite at 0 frequency, so if we collect something at 0Hz we are in trouble. So we add positive and negative deltas to acquire more times the signal (10 times instead of 5), but at the same time to remove the zero frequency.

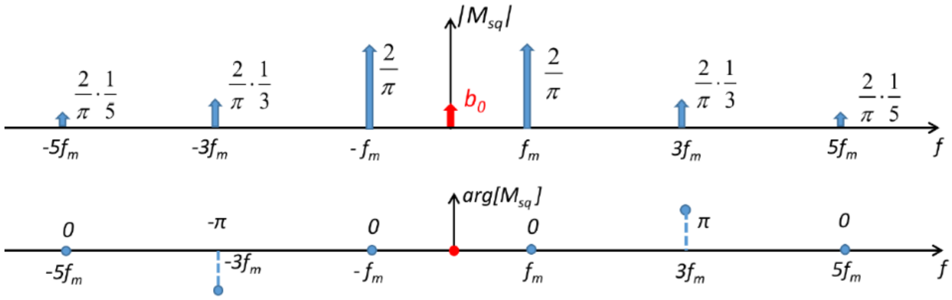
Starting from this idea, we have some problems. In fact, if we compute the SNR, we find a factor 2 in the lower part of the ratio, so it seems that we are collecting twice the noise. Indeed, we have a factor 2 on the noise, but we are doubling only the WN, while without the LIA we are not doubling it, but we are also not doubling the 1/f with LIA.

The second problem was that, in order to create the LIA, we need an analog multiplier, which is a nightmare, since it might add distortion. As soon as we make a distortion, we are also distorting the reference and we are adding lines in the spectrum, so also collecting noise from these lines.

So the idea, since the analog multiplier is a problem, is to make the same 'job' with a square wave reference, because in all the formulas there is no reference on the shape of the reference.

<u>Using a square waveform allows us to avoid the multiplier. In fact, multiplying by a square waveform means multiplying by B or -B, so we need just a couple of amplifiers and a switch</u>.

Fourier transform of a real square wave



$$M_{sq}(f) = \sum_{k=0}^{\infty} \frac{b_{2k+1}}{2}\left[\delta(f - f_{2k+1}) + \delta(f + f_{2k+1})\right] \quad \text{with} \quad b_{2k+1} = \frac{(-1)^k}{(2k+1)} \cdot \frac{4}{\pi}$$

$k = 0, 1, 2, 3 .....$ and $f_{2k+1} = (2k+1) f_m$

BEWARE: In cases where the squarewave has non-zero mean value (e.g. slight asymmetry in amplitude and/or duration of positive and negative lobes) it has also a DC component with amplitude b_0 given by the ratio of the mean value to the peak amplitude (i.e. by the relative unbalance of positive and negative area)

We can notice that we are collecting more noise than required, fm, 3*fm, 5*fm, …, but also more signal. If we have a small offset, we might have a problem that depends on how much is big the delta at 0Hz.

If we compare it with the sinusoidal reference, it seems that we are collecting more noise. Hence for sure we will obtain a lower SNR with this reference than with the sinusoidal one. The bandwidth is exactly the same, because we are using the same LP filter, but with the sinusoidal we were collecting noise only on fm and -fm.

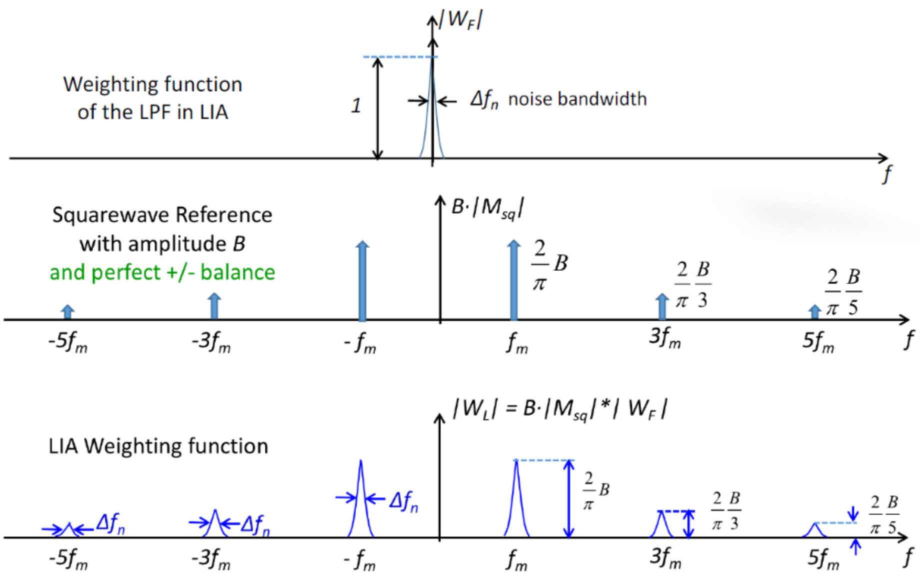
Let's compute the SNR in this case and if it is possible to make a workaround.

Frequency domain

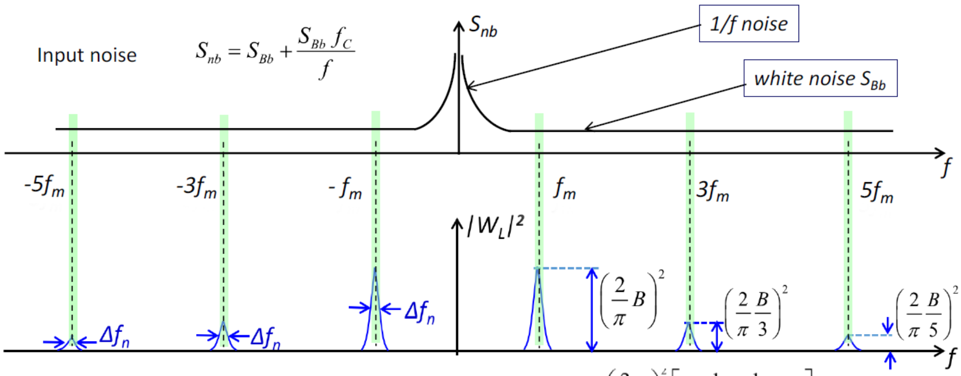
The w_f definition tells us that we compute the modulus and we make the integral from -inf to +inf of the Fourier of the signal times the Fourier of the w_f calculated in -f.

Time domain

We are making the product of a cosinusoidal times the square waveform, which is between -B and +B, so we will have B*A, and then the LP filter makes the average. We get (2/pi) * B*A.



Every time we have a line, we collect a BW of delta(f) of WN, we don't have 1/f. Then we have to sum all the values at all the lines.



Output Noise $\overline{n_{yL}^2} = \int_{-\infty}^{\infty} S_{nb}(f)|W_L(f)|^2 df = 2 \cdot S_{Bb}\Delta f_n \left(\frac{2}{\pi}B\right)^2 \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$

The factor $\left[1+(1/3)^2+(1/5)^2+\dots\right] = \pi^2/8 \cong (1{,}11)^2$ represents the enhanced noise due also to the higher passbands at the harmonic frequencies

$$\overline{n_{yL}^2} = 2S_{Bb}\Delta f_n \left(\frac{2}{\pi}\right)^2 B^2 \frac{\pi^2}{8} = B^2 S_{Bb}\Delta f_n$$

No signal is collected in these passbands. Therefore, the S/N is reduced with respect to the case of sinusoidal reference, but the reduction is moderate.

SNR WITH SINUSOIDAL SIGNAL AND PERFECT SQUAREWAVE REFERENCE

Output Signal $s_y = \frac{2}{\pi} B \cdot A$ (for sinusoidal input signal in phase)

Output Noise $\overline{n_{yL}^2} = S_{Bb}\Delta f_n B^2$

so that $\left(\frac{S}{N}\right)_{L,sqw} = \frac{s_y}{\sqrt{\overline{n_{yL}^2}}} = \frac{A}{\frac{\pi}{2}\sqrt{S_{Bb}\Delta f_n}}$

which in comparison to the result obtained with sinusoidal reference

$$\left(\frac{S}{N}\right)_{L,\sin} = \frac{A}{\sqrt{2S_{Bb}\Delta f_n}}$$

is just moderately lower

$$\left(\frac{S}{N}\right)_{L,sqw} = \frac{\sqrt{2}}{\frac{\pi}{2}}\left(\frac{S}{N}\right)_{L,\sin} \cong \frac{1}{1{,}11}\left(\frac{S}{N}\right)_{L,\sin}$$

We will now deal with another case often met in practice: the signal to be measured is a squarewave in phase with the squarewave reference

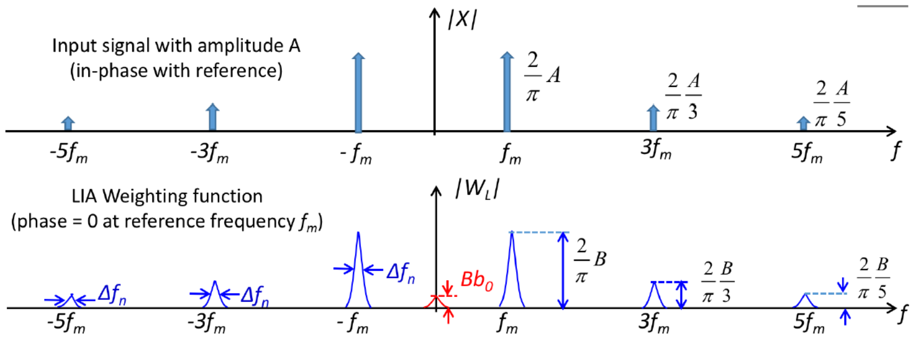
We can see that the SNR is a factor 1.11 lower than in the case of a sinusoidal waveform. It is not strange because we are more noise.
So there is a trade off here. On one side I don't have to use an analog multiplier, on the other side I'm reducing the SNR. How can we get high SNR?

This solution is perfect from the cost and real implementation standpoints, but we are paying SNR because the problem is that we are collecting, together with the signal, also the noise where there is no signal. So signal two times and noise an infinite number of times.

The first solution was to remove the BW where there is no signal, so we are collecting noise where the signal is, but if we don't want to remove the BW where there is no signal, we have to see if it is possible to put the signal where we collect the noise.

Instead of a sinusoidal we use a square wave signal.

LIA WITH SQUAREWAEV SIGNAL AND REFERENCE



Output signal

$$s_y = \int_{-\infty}^{\infty} X(f) W_L(-f) df = 2 \cdot \left[\frac{2}{\pi} A \cdot \frac{2}{\pi} B\right] \cdot \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right] = 2 \cdot \left[\frac{2}{\pi} A \cdot \frac{2}{\pi} B\right] \frac{\pi^2}{8} = AB$$

NB1: This is easily verified in time, since: LIA output $y(t)$ = time average of $z(t)=x(t) \cdot B\, m(t)$

NB2: As concerns the **output noise**, it has already been discussed

The signal has now a known shape, exactly the same shape of the reference, with a value A. We want to assess the SNR.
For the signal is the integral from -inf to +inf of Fourier t.f. of the signal times the w_f calculated at -f, according to the image above.
As for the noise, it is the same as before, since the reference tells us where we are collecting the noise and it is still a square wave.

Putting all together we have the following.

Output Signal $\quad s_y = B \cdot A \quad$ for squarewave input signal in phase

Output Noise $\quad \overline{n_{yL}^2} = S_{Bb} \Delta f_n B^2$

so that $\quad \left(\frac{S}{N}\right)_{L,sqw} = \frac{s_y}{\sqrt{\overline{n_{yL}^2}}} = \frac{A}{\sqrt{S_{Bb} \Delta f_n}}$

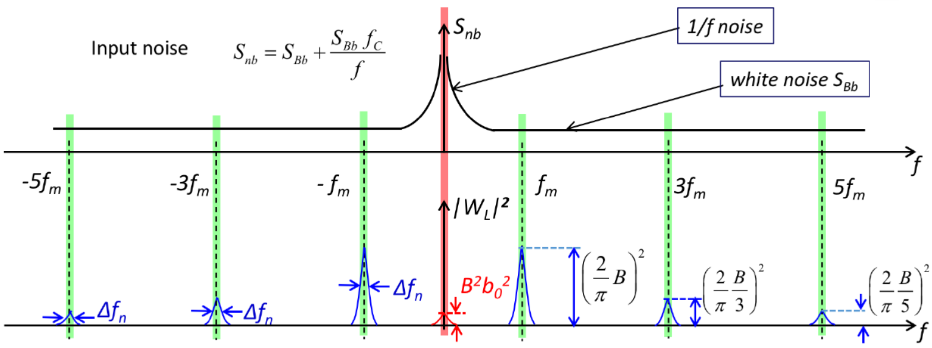
However, for equal amplitude *A* the squarewave signal has double power and correspondingly higher S/N

It is the SNR we got using a non-modulated signal but with a spectral density that is only the one of the WN, without 1/f, removing also the factor 2.

So it seems that this solution is better than the LIA with a sinusoidal reference and signal. It seems impossible to get this result.
Delta(f) is the BW of the LP filter I'm using, which depends on the real signal before the modulation, so it's a data. Sb is the spectral density of the WN (with a sinusoidal we have a factor 2). A is the amplitude of the signal we want to obtain, the information we want to get. A modulated signal with amplitude A has a power A^2/2, which is half the power A^2 of a constant value signal before the modulation.

So if we change the way in which we modulate the signal from waveform to sinusoidal without touching the power, we are loosing in SNR. But if we are limited by the power dissipated by the sensor, if we use a square waveform the power is A^2, but in the sinusoidal is A^2/2, so we can double the power supply and dissipate the same. If we dissipate the same power we have exactly the same SNR.

NOISE THROUGH LIA WITH IMPERFECT SQUAREWAVE REFERENCE



A squarewave with non-zero-mean generates a spurious band at f=0 with additional noise

$$\overline{n_{yL0}^2} = \widehat{S_{nb}} \Delta f_n B^2 b_0^2$$

Because of the 1/f noise, the mean density $\widehat{S_{nb}}$ in the band can be very high $\widehat{S_{nb}} \gg S_{Bb}$ so that even with small spurious band b_0<<1 the added noise $\overline{n_{yL0}^2}$ can be comparable to the basic term $\overline{n_{yL}^2}$ or even larger

$$\boxed{\frac{\overline{n_{yL0}^2}}{\overline{n_{yL}^2}} = \frac{\widehat{S_{nb}}}{S_{Bb}} b_0^2}$$

Let's suppose we have a light signal that we modulate on-off. Do we still have the delta at 0Hz?
The average of the signal is 50%. So there is the b0 or not? b0 is the DC component of the w_f. In the time domain, the w_f of the LIA is the reference signal times the LP filter. The LP gives us the BW, the line is fixed by the reference, so b0 is in the reference if I have it.
In our application, the reference is something that gives us the sync with the signal. If the signal has 1/0, the reference can be 1/-1, since we are interested in the synchronization. In our case is not important the DC component of the signal, because the signal doesn't collect any BW on the noise, it is the w_f that has to avoid the DC component.

So with a light source I modulate a laser with a sinusoidal waveform placing an offset and then over the offset I modulate, because we cannot have negative light. Offset is not a problem if the same sinusoidal is used as a reference but with no offset. It is the reference that must have no offset.

Summary and comparison

	SINUSOIDAL Reference	SQUAREWAVE Reference
SINUSOIDAL Signal amplitude A power $P=\frac{A^2}{2}$ A_{min} minimum measurable amplitude (at S/N=1)	$\frac{S}{N}=\frac{A}{\sqrt{2}\sqrt{S_{Bb}\Delta f_n}}=\frac{\sqrt{P}}{\sqrt{S_{Bb}\Delta f_n}}$	$\frac{S}{N}=\frac{A}{\frac{\pi}{2}\sqrt{S_{Bb}\Delta f_n}}=\frac{\sqrt{P}}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}}$
	$A_{\min}=\sqrt{2}\sqrt{S_{Bb}\Delta f_n}=1{,}41\sqrt{S_{Bb}\Delta f_n}$	$A_{\min}=\frac{\pi}{2}\sqrt{S_{Bb}\Delta f_n}=1{,}57\sqrt{S_{Bb}\Delta f_n}$
SQUAREWAVE Signal amplitude A power $P=A^2$ A_{min} minimum measurable amplitude (at S/N=1)	$\frac{S}{N}=\frac{A}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}}=\frac{\sqrt{P}}{\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}}$	$\frac{S}{N}=\frac{A}{\sqrt{S_{Bb}\Delta f_n}}=\frac{\sqrt{P}}{\sqrt{S_{Bb}\Delta f_n}}$
	$A_{\min}=\frac{\pi}{2\sqrt{2}}\sqrt{S_{Bb}\Delta f_n}=1{,}11\sqrt{S_{Bb}\Delta f_n}$	$A_{\min}=\sqrt{S_{Bb}\Delta f_n}$

REFERENCE PHASE ADJUSTMENT

In order to be useful as **reference** for measuring with a LIA a given periodic signal, the **essential necessary features** of an auxiliary signal are:

1) fundamental **frequency identical** to the signal
2) **constant phase difference φ** with respect to the signal.
(NB: not necessarily φ=0, but it is necessary that φ = constant !)

If the auxiliary signal has high and constant amplitude, negligible noise and clean waveform (free from harmonics), it can be directly adjusted to φ=0 with a phase-shifter filter and supplied to the multiplier as reference waveform.

- An adjustable phase-shifter is currently included in LIAs for re-phasing the reference. The phase adjustment can be controlled manually by observing the output signal amplitude, which is maximum when φ=0.
- Many LIA's besides the adjustable phase shifter include an additional filter, which gives phase shift φ_a switchable from $\varphi_a=\pi/2$ to $\varphi_a=0$. Setting $\varphi_a=\pi/2$, when φ=0 is reached the signal is in quadrature and the output is zero. Notice that observing the output signal while φ is varied it is easier to identify when it reaches zero rather than when it reaches the maximum. After the adjustment to φ=0, the additional filter is switched back to $\varphi_a=0$ and the LIA is ready to operate.

A cosinusoidal has a frequency and a phase. Normally we use the phase = 0 because the important thing is that it remains constant, otherwise we are changing the modulation.

By exploiting the a well known trigonometric equation

$$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$$

in cases with sinusoidal signal and sinusoidal reference

$$x(t)=A\cos(2\pi f_S t)$$
$$m(t)=B\cos(2\pi f_m t+\varphi_m)$$

the result is directly obtained, since $f_S=f_m$

$$\boxed{y(t)=x(t)\cdot m(t)=\frac{AB}{2}\cos[\varphi_m]+\frac{AB}{2}\cos[2\pi(2f_S)t+\varphi_m]}$$

PHOTONS

SPECTRAL RANGES

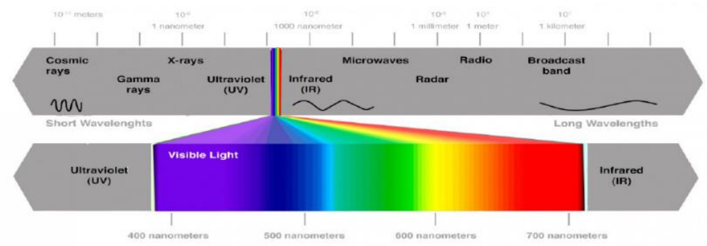
Changing the sensor we might change the signal we receive and the work to be done from the signal recovery standpoint. We want to understand the performances of light sensors.
As for light, we are interested in the **speed of light c**. The near infrared range is of interest because of the sensor; at this moment, the best sensors are developed in the visible range, because still dominated by the silicon technology.

- Light = electromagnetic waves with frequency ν and wavelength λ
  propagation speed (in vacuum) $c = 2{,}998 \cdot 10^8\ m/s$

$$\boxed{c = \lambda \nu}$$

- **Spectral ranges:**

$\lambda < 400nm$	Ultraviolet (UV)
$400nm < \lambda < 750nm$	Visible (VIS)
$750nm < \lambda < 3\ \mu m$	Near-infrared (NIR)
$3\ \mu m < \lambda < 30\ \mu m$	Mid-infrared (MIR)
$30\ \mu m < \lambda$	Far-infrared (FIR)



PHOTON ENERGY AND MOMENTUM

We want to detect the energy associated to a single photon, this is the signal we are interested about. We move from the energy in Joule to the electron voltage, which is the voltage which, multiplied by the charge of the electron, gives us the energy.

Photon: quantum of electromagnetic energy

$E_p = h\nu$ quantum energy (Planck's constant $h = 7{,}6 \cdot 10^{-34}\ J{\cdot}s$)

Rather than E_p in Joules, the electron-voltage V_p is employed:

$E_p = q\,V_p$ (electron charge $q = 1{,}602\ 10^{-19}\ C$ V_p in Volts or electron-Volts eV)

from $E_p = qV_p$ we get $$V_p = \frac{hc}{q}\frac{1}{\lambda}$$

universal constant $hc/q = 1{,}2398 \cdot 10^{-6}\ m{\cdot}V \approx 1{,}24\ \mu m{\cdot}V$

X $$\boxed{V_p = \frac{1{,}24}{\lambda}}$$ with V_p in Volts and λ in µm

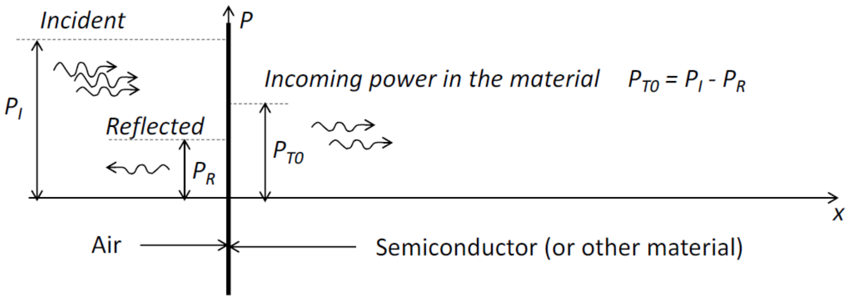
$400nm < \lambda < 750nm$	VIS range	$3{,}10\ eV > V_p > 1{,}65\ eV$
$750nm < \lambda < 3\mu m$	NIR range	$1{,}65\ eV > V_p > 0{,}41\ eV$

The eV gives us information about the sensors we can use. In the case of silicon, the gap of energy is 1.12 eV; if we know that 1.1 eV is the gap of silicon, we can detect photons only if its energy is higher than 1.1 eV. Formula x is the one to remember.

In the visible range we don't have problem with silicon, while in the near infrared we can detect photons up to 1.1 um of wavelength. So we stop in the visible range as field of interest because we want not only to detect photons, but also to detect them with high efficiency.

REFLECTION AND ABSORPTION OF PHOTONS

We want to create an electrical signal from light. The first problem, often neglected, is the absorption of light. Indeed, the light has to arrive inside the sensor, and it could be reflected on the surface. The reflective index of air is 1, but of any other material is not 1, and if the reflective index changes, at the interface we have reflection. If light is reflected, there is no way to absorb it.
To solve this, we can use an anti-reflection coating, modulating the reflection index in a smooth way to avoid the reflection. It is very important and it must be chosen carefully depending on the application, otherwise we could waste 20-30% of the light.



At the surface strong discontinuity of the refraction index *n*, from *n* = 1 for air to *n* >1 for semiconductor: e.g. for silicon it is about n ≈ 3,4 and depends on the wavelength.
This discontinuity gives a **high reflection coefficient *R***

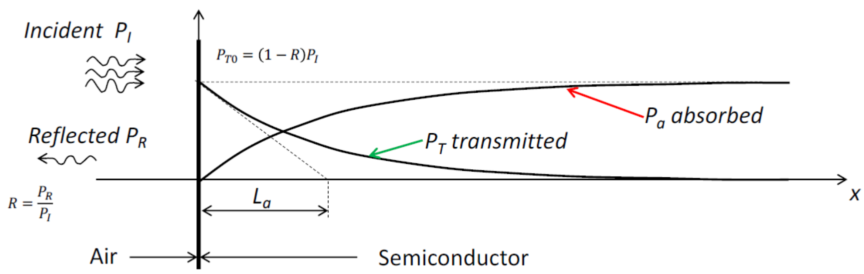
$$R = \frac{P_R}{P_I}$$ (e.g. for silicon *R* > *0,4* wavelength dependent)

Anti-reflection coating: deposition on the reflecting surface of a sequence of thin dielectric material layers with progressively decreasing *n* value. It provides a **gradual decrease** of the n value from semiconductor to air and such a smoother transition reduces the reflection

Absorption of photons

What happens inside the detector?

If the light is not so high from the power point of view, the absorption is quite linear (moderate or low Pt).



For moderate or low P_T the absorption in *dx* is proportional to P_T (linear optic effect)

$$-dP_T = \alpha P_T dx = P_T \frac{dx}{L_a}$$

α = **optical absorption coefficient**
$L_a = 1/\alpha$ = **optical absorption depth**

The optical power transmitted to position *x* is

$$P_T = P_{T0} \exp(-\alpha x) = P_{T0} \exp\left(-\frac{x}{L_a}\right)$$

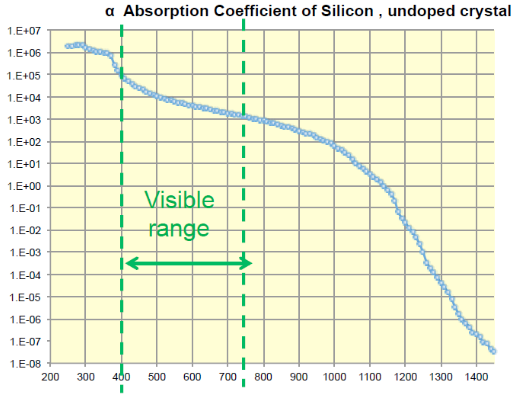
The optical power absorbed from 0 to *x* is

$$P_a = P_{T0} - P_T = P_{T0}(1 - e^{-\alpha x}) = P_{T0}\left(1 - e^{-\frac{x}{L_a}}\right)$$

Alpha is the optical absorption coefficient, L is the optical absorption length → alpha = 1/L.
If we solve the equation, we get the classical exponential, and L gives us the tau of the exponential. L depends on the wavelength, so on the application in our case. Our goal is to absorb all the light. If the wavelength is big, we have to increase L; however, thermal generation is proportional to the length, so if we increase the dimensions we are also increasing the noise.

For a given material the optical absorption **STRONGLY** depends on the **WAVELENGTH**.

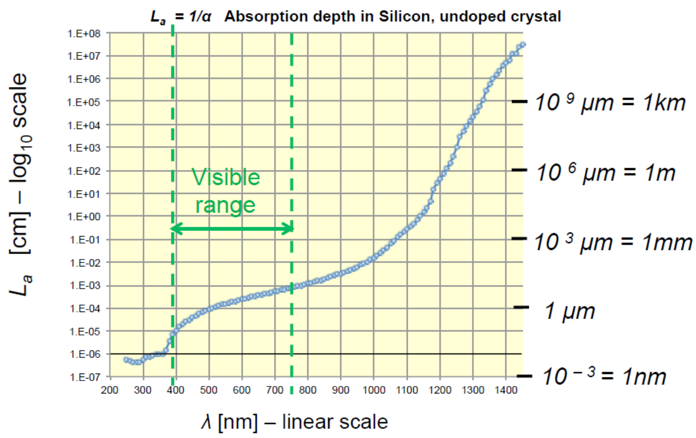
Typical example: Silicon absorption coefficient



The problem, looking at the graph, is that alpha changes from 10^7 to 10^-8, it is a huge change. We can take another graph that is the same but flipped, so it is the absorption length L.

For a given material the optical absorption **STRONGLY** depends on the **WAVELENGTH**.

Typical example: Silicon absorption depth



NB: over the visible range L_a varies with λ by two orders of magnitude!!

1/alpha is the absorption length of the exponential decay time; if we want to absorb all the light we need 4 to 5 tau, so we can retrieve the thickness of the detector to collect all the light. The important values are 400 nm, the starting point of the visible range and blue light (corresponding to L = 0.1 um), 5 nm, which is the green (1 um of L), 800 nm, which corresponds to 10 um of L.

THERMAL PHOTODETECTOR PRINCIPLES

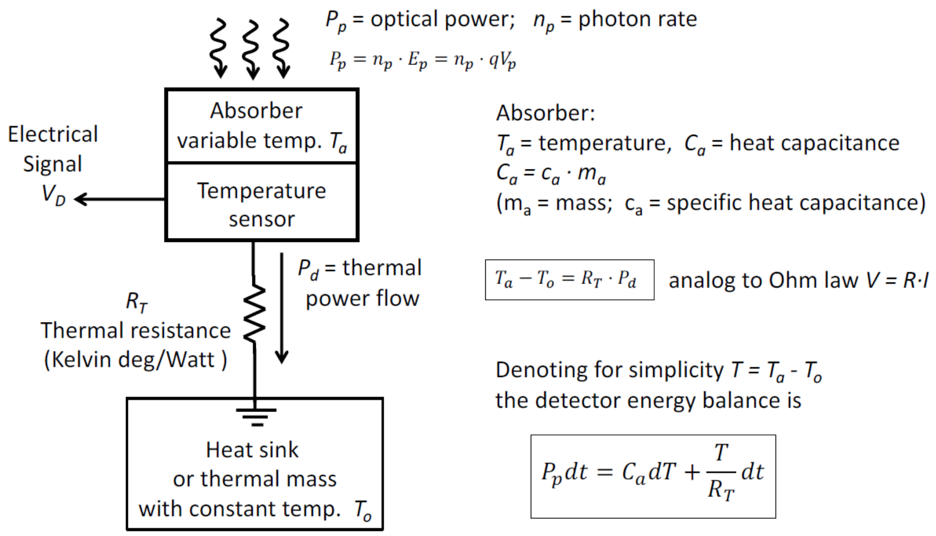
- A principle for detection of light signals is to **employ their energy simply for heating a target and measure its temperature rise *ΔT***. Detectors relying on this principle are called «**Thermal Photodetectors**» or «**Power Detectors**»
- Main advantage: very **wide spectral range**. Since photons just have to be absorbed for contributing to the detection, the range can be extended far into the infrared.
- Main drawback: sensitivity is inherently poor, because a high number of absorbed photons is required for producing even small variations of temperature ΔT in tiny target. For instance: ≈$\mathbf{10^{15}}$ blue photons are required for heating by **ΔT=0,1 K** a water droplet of ≈ **1mm** diameter (*blue photons at λ=475nm have V_p = 2,6 eV; water has specific heat capacity c_T = 4186 [J/Kg·K]= 2,6 ·10^{22} [eV/Kg·K] and the mass is 1mg*)
- The dynamic response is inherently slow, because thermal transients are slow. Thermal detectors are mainly suitable for measurement of steady radiation.

We have to understand how from the absorption of light we can generate a signal and which is the signal we generate. The first sensor we use is the thermal photodetector, which is no more used nowadays. Light has some power, which creates an increase of temperature in the material absorbing the light. The idea is to measure the increase in temperature. All the light can contribute to the increase of temperature, doesn't matter if visible or infrared light, all can change the temperature. So with this detector we can detect any kind of light.

The problem is that, to maximize the change in temperature, we need to reduce the amount of material; even with a small amount of material, we need a lot of photons to change the temperature of 0.1°C. Nevertheless, the sensitivity is not the most important problem.

Principle of thermal photo-detector

We measure the temperature of the material with respect to the ambient temperature (heat sink). In the middle we have a thermal resistance.



From the energy balance $P_p dt = C_a dT + \frac{T}{R_T} dt$

we get $\frac{dT}{dt} = \frac{P_p}{C_a} - \frac{T}{R_T C_a}$ and in Laplace transform $sT = \frac{P_p}{C_a} - \frac{T}{R_T C_a}$

The detector transfer function from optical power to measured temperature thus is

$$\text{x} \quad \boxed{T = P_p R_T \frac{1}{1 + sR_T C_a}}$$

- The steady state response (the steady $T = P_p R_T$ obtained with steady P_p) increases as the thermal resistance R_T is increased
- The dynamic response is a single-pole low-pass filter with characteristic time constant $\tau_a = R_T C_a$: as R_T is increased, the bandlimit $f_T = 1/2\pi\, R_T C_a$ is decreased
- For improving the high-frequency response without reducing the steady response it is necessary to **reduce the heat capacitance $C_a = c_a \cdot m_a$**. This implies that
  a) absorber materials with small specific heat capacitance c_a are required
  b) the absorber mass m_a should be minimized.
- Remarkable progress has been indeed achieved in thermal detectors with modern **technologies of miniaturization and integration (of absorber, temperature sensor, etc.)** that make possible to fabricate also multipixel arrays of thermal detectors

The formula x is a LP filter, and this is the real problem of this filter, since it acts on light as a LP filter. The only way to speed up this sensor is to reduce the capacitance, so to reduce the mass; but if so, we need to find a temperature sensor able to read a small amount of temperature. Nowadays, the speed of the sensor is very important as well as the SNR.

RADIANT SENSITIVITY OR SPECTRAL RESPONSIVITY

The radiant sensitivity is the electrical output voltage divided by the optical power of the detector, for this kind of sensors. It is independent on the wavelength of the radiation light.

- Thermal detectors transduce the optical power P_P in an electrical output signal V_D of the temperature sensor (voltage signal of thermoresistances in Bolometers and of thermocouples in Thermopiles).
- The basic quantitative characterization of the performance of the detector is given by the **Radiant Sensitivity** (also called Spectral Responsivity) S_D, defined as

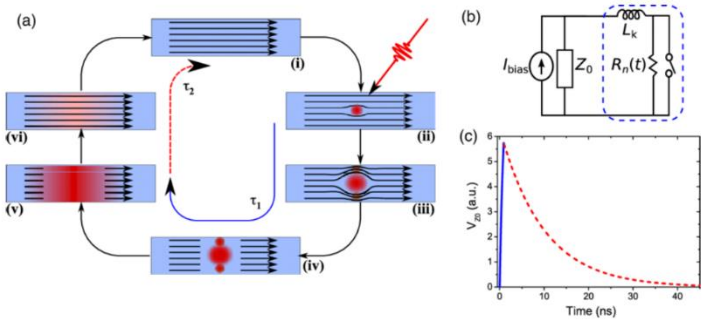
$$S_D = \frac{\text{electrical output voltage [in V]}}{\text{optical power on the detector sensitive area [in W]}}$$

- For a given absorbed power the detector is heated at a given level, **independent of the radiation wavelength λ**. Therefore, uniform S_D would be obtained at all λ if the reflection and absorption were constant, independent of λ.
- Variations of reflection and absorption vs λ are kept at moderate level with modern absorber technologies. Fairly **uniform S_D is achieved** over fairly wide wavelength ranges, extended well into the infrared spectral region.

SUPERCONDUCTING NANOWIRE

The idea is that we take a piece of material and we cool down it at very low levels, 0.4 K. At this point, we switch in the superconductive regime; in this regime, the material has no resistance and infinite conductivity. As soon as we absorb one photon, which has a certain energy, we change the temperature of the material, which it is at superconductive level, so it is enough a small change in temperature to break the superconductivity, and the resistance of the material is no more zero, it increases to the value of the material. So **in this way we can detect every single photon**.
As far as we absorb the photon, after a while the material returns to the previous state because the heat is dissipated. So we can neglect a single photon, but with which precision? Up to some ps.

Some problems are related to the cost of the system and the dimensions of these materials, which are very small, nm, and focusing the light on few nanometers is difficult.



QUANTUM PHOTODETECTOR PRINCIPLES

We need to create something from the commercial point of view. The idea is that we use the energy of the single photon or light not to change the temperature of the material but to create a carrier. We use the photoelectric effect to product a carrier, we create electrons directly from light. We have two types of detectors: **vacuum tubes** and **semiconductor devices**. They share the same working principle, and nowadays vacuum tubes are still used a lot in many applications, because we can create very large devices with low noise, while this is not possible with semiconductor devices.

- A different principle for the detection of light signals is to exploit **photo-electric effects for producing directly an electrical current** in the detector. The energy of the absorbed photons is used for generating free charge carriers, which constitute the elements of the detector current.

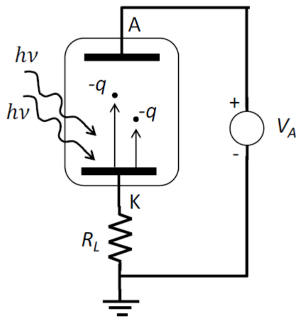
- Detectors relying on this principle are called «**Quantum Photodetectors**» or «**Photon Detectors**»

- Photon Detectors can be vacuum-tube or semiconductor devices

Principles of Quantum Photodetectors

Vacuum tube

Let's take a vacuum tube made of glass and we have an electrode on one side, the cathode where photons impinging can create electrons, which are emitted and are in the middle of an electric field due to the reverse voltage and are collected at the anode.

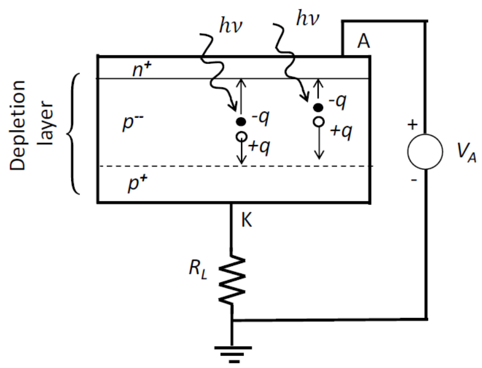


Vacuum-Tube detector devices:
Photo-Tubes or Photo-Diodes

- An electrode (cathode K) in a vacuum enclosure receives the photons
- By **photo-electric effect** the cathode emits electrons in vacuum.
- The **electrons are drawn by the electric** field to another electrode biased at higher potential (anode A)
- **Current flows through the terminals** (photocathode and anode).

Semiconductor detectors

The photodetector is the silicon photodetector, so the pn junction. The photon creates a electron/hole pair and due to the electric field we have a current. The idea is the same of the vacuum tube, but in the case of the vacuum tube the detector is bulky and fragile (made of glass)



Semiconductor detector devices:
Photo-Diodes

- Photons impact on a reverse-biased p-n junction diode
- The **absorbed photons raise electrons from valence band to conduction band** of the semiconductor, thereby generating free electron-hole pairs.
- The free carriers generated in the zone of high electric field **(the depletion layer)** are drawn by the junction electric field (the electrons to the n-terminal and the holes to the p-terminal)
- Current flows through the terminals.

EFFICIENCY

- Quantum photodetectors **transduce optical signals in electrical current signals by collecting the free electrons** generated by the photons of the optical radiation.
- The basic quantitative characterization of the performance of the detector is given by the **Quantum Detection Efficiency** (or Photon Detection Efficiency) η_D defined as

$$\eta_D = \frac{\text{number of photogenerated electrons (or electron-hole pairs)}}{\text{number of photons reaching the detector}} = \frac{N_e}{N_p}$$

- However, since in many engineering tasks the focus is on the transduction from optical power to electrical current, the **Radiant Sensitivity** S_D is often employed also for quantum photodetectors, defined as

$$S_D = \frac{\text{electrical output current [in A]}}{\text{optical power on the detector sensitive area [in W]}} = \frac{I_D}{P_L}\left[\frac{A}{W}\right]$$

The photon detection efficiency is the ratio between the number of photogenerated electrons (or e/h pairs) divided by the number of photons reaching the detector.

So let's assume to have some photons with a wavelength lambda arriving with a rate n_p on a quantum detector. We can compute the optical power that is n_p times the energy of each photon. Hence we have an output current that is the multiplication of the charge of each electron times the rate of the produced current.

Then, the radiant sensitivity is the ratio between the current and the optical power.

Photons of wavelength λ arriving with steady rate n_p on a quantum detector convey an optical power P_L

$$P_L = n_p h\nu$$

the electrons (or e-h pairs) photogenerated in the detector with steady rate n_e produce a current

$$I_D = n_e q$$

The Radiant Sensitivity is

$$S_D = \frac{I_D}{P_L} = \frac{n_e}{n_p} \cdot \frac{q}{h\nu} = \frac{n_e}{n_p} \cdot \frac{\lambda}{\frac{hc}{q}}$$

and since $\eta_D = n_e / n_p$

$$\boxed{S_D = \eta_D \cdot \frac{\lambda}{\frac{hc}{q}} = \eta_D \cdot \frac{\lambda[\mu m]}{1{,}24}}$$

*We see that the Radiant Sensitivity of the quantum detectors **intrinsically depends on the wavelength λ**, that is, **even with constant quantum efficiency η_D**. This occurs because a given optical power P_L corresponds to different photon rates n_p at different wavelengths λ*

Differently from the bolometer, radiant sensitivity strongly depends on the wavelength, so if we maintain the same power but we change the wavelength we are changing the energy of each photon so the number of photon, so the output current. This is not a big issue.

PHOTON STATISTICS AND NOISE

- The optical radiation is composed of photons arriving randomly in time; the photon number N_p in a given time interval T is a statistical variable with mean $\overline{N_p}$ and variance $\sigma_p^2 = \overline{N_p^2} - \left(\overline{N_p}\right)^2$
- **The random fluctuations of the photons are the noise already present at optical level.** This optical noise can be due to a background photon flux and to the actual desired optical signal.
- In most cases the photon statistics is well approximated by the Poisson statistics, so that it is

$$\boxed{\sigma_p^2 = \overline{N_p}}$$

- The optical power arriving to the detector is composed of quanta with energy $h\nu$ arriving randomly at rate n_p. It is the analog at optical level of a shot electrical current: the mean optical power is $P_p = n_p h\nu$ (analog to $I_e = n_e q$) ; the shot optical noise has unilateral spectral density S_p (analog to $S_i = 2qI_e$)

$$\boxed{S_p = 2h\nu P_p = 2\frac{hc}{\lambda} P_p}$$

- Note that for a given optical power P_p the shot noise density decreases as the wavelength λ is increased

The arrival of photons can be approximated with a Poisson statistic, so the variance of the process is equal to the average value.
From the noise standpoint, we can say that the optical power arriving to the detector is composed by quanta with energy hv, since each photon arrives with energy hv, with a random rate n_p.

This is the exact same situation that we have for the shot noise with the current, so we get the same unilateral spectral density of the shot noise, in functional form.

CURRENT SIGNALS OF QUANTUM PHOTODETECTORS

We can start by studying the delta response of the system. We have always a cut off at high frequency; also if we have a very fast shot of light, the output is not a delta, the output is still fast but with longer duration.
Firstly, we define some parameters that help us to define the situation, such as the single electron response: the response of our detector to a single photon. If we don't have a single photon we can use linear superposition and reconstruct the response of multiphoton as a linear superposition of a single electron response.

- In the transduction of optical signals to current signals by Quantum Photodetectors the dynamic response has a cut-off at high frequency. Ultrafast optical pulses are transduced to current pulses that are still fast, but have longer duration.
- The response to a multi-photon optical signal is the linear superposition of the elementary responses to individual photons. The response to a single photon is also called **Single-Electron-Response SER** because a photon generates just one free electron (or one electron-hole pair).
- It is simply **wrong** to consider the SER a **δ-like current pulse** occurring at the time where the photogenerated charge carrier impacts on the collector electrode. The carrier **induces** a charge in the collector electrode **before** reaching it; the induced charge varies with the carrier position, so that current flows during all the carrier travel in the electric field.
- The waveform of the current signal is obtained by taking the derivative of the charge induced on the collector electrode as a function of time. To compute this charge is an electrostatic problem not easy to solve in general. However, the mathematical treatment can be remarkably simplified by preliminarily computing the motion of the charge carriers and exploiting then the **Shockley-Ramo theorem**.

The Shockley-Remo theorem helps us in understanding which is the output in current

Shockley-Ramo theorem

The output current due to an electron traveling towards the collector electrode can be obtained by applying the Shockley-Ramo theorem in three steps

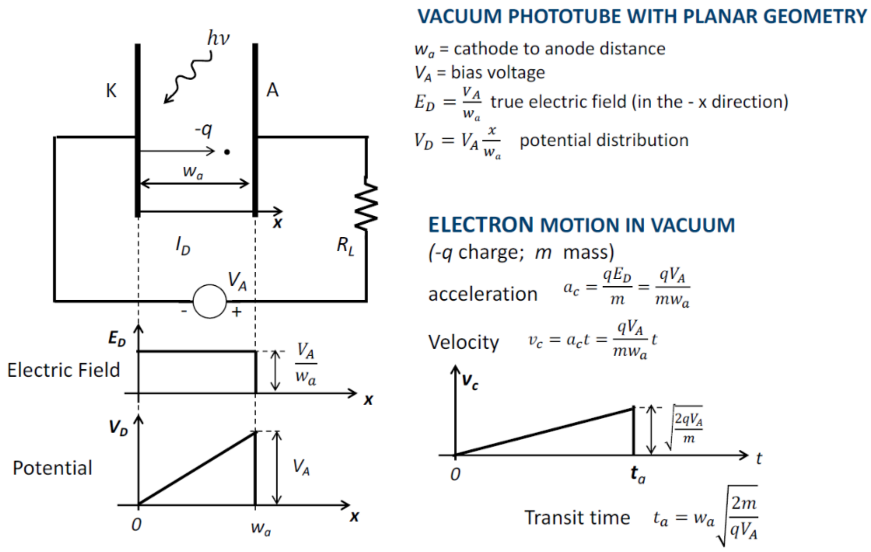
1. The motion of the electron must be computed; i.e. the trajectory and the velocity v_c at every point of it must be known
2. A reference electric field E_v must be computed, which is the field that would exist in the device (in particular along the electron trajectory) under the following circumstances:
   electron removed
   output electrode raised at unit potential
   all other conductors at ground potential
3. The **Shockley-Ramo theorem** states that the current i_c that flows at the output electrode due to the electron motion can be simply computed as

$$\boxed{i_c = q\overrightarrow{E_v} \bullet \overrightarrow{v_c} = qE_{vc}v_c}$$

where • denotes scalar product and E_{vc} is the component of the field $\overrightarrow{E_v}$ in the direction of the velocity $\overrightarrow{v_c}$

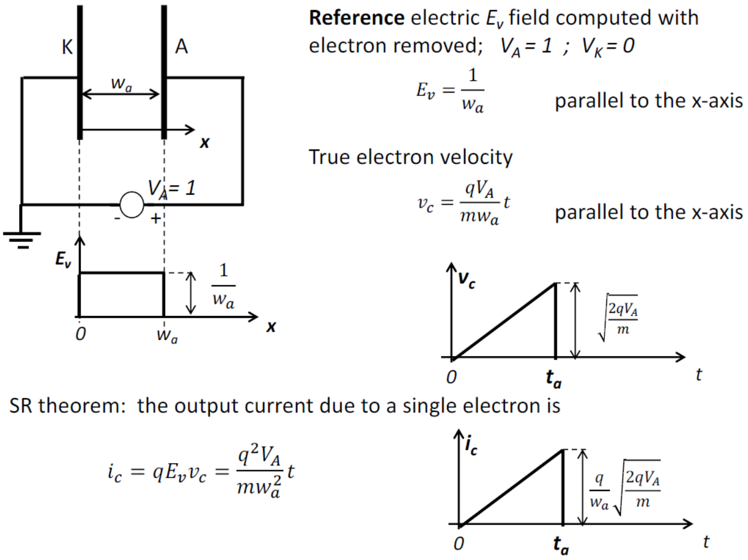
Firstly we study the motion of the electron in terms of trajectory and velocity. Then we have to use a reference electric field with the electron removed, the output electrode raised to unitary potential and all the other conductors at ground level.

Carrier motion in a phototube (PT)



The potential distribution is linear. The trajectory of the electron is linear from the cathode to the anode. The transit time is the time at which we reach the maximum speed, that is the speed we have at the anode.

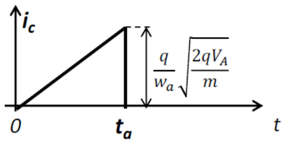
Then we have to ground all the conductors except for the output collector anode that has to be set to a potential of 1. Since Va = 1 and cathode is to ground, the electric field is 1/w_a.



Single electron response (SER)

In a phototube with planar geometry the **single electron response (SER)** is a pulse with triangular waveform

$$i_c = qE_v v_c = \frac{q^2 V_A}{m w_a^2} t \qquad (0 \le t \le t_a)$$



The frequency response is the Fourier transform of the SER pulse, which has a high frequency cutoff inversely proportional to the pulse width.
The pulse width is set by the transit time t_a of the electron from cathode to anode

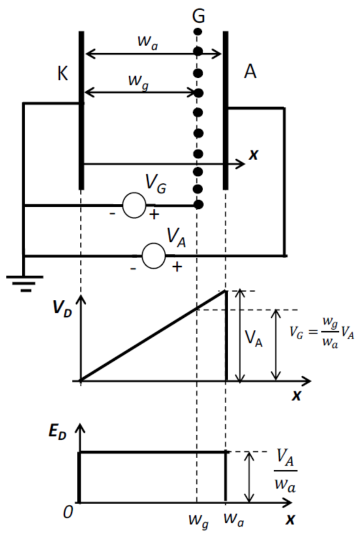
$$t_a = \sqrt{2\frac{m}{q}} \cdot \frac{w_a}{\sqrt{V_A}} = 3{,}37 \cdot 10^{-6} \frac{w_a}{\sqrt{V_A}}$$

Typical values for phototubes are around *w = 1cm = 0,01m* and V_A= 100V, which correspond to transit time around $t_a \approx$ ***3,3 ns***

It is a detector with an extremely high bias voltage, 100V, and we have a transit time of 3.3ns, which is almost 300MHz in the frequency domain. We can further increase the speed of the PT, reducing the transit time. We have to increase the bias voltage, which is however leading to an increase in power dissipation. Hence we can reduce the width of the detector, but in doing so we might increase the stray capacitance. So the idea is to use a grid.

SCREENED-ANODE PT

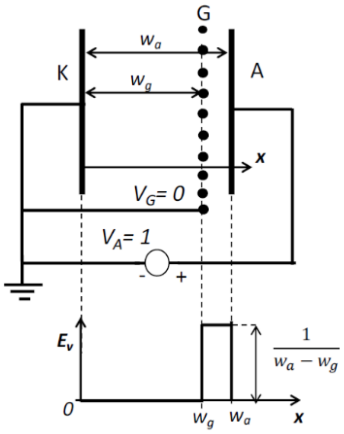
We use the grid to reduce the SER pulse. The idea is that the grid acts as an electrostatic screen.
Since I'm adding a grid, am I changing the behaviour of the detector? Not necessarily, if I put the grid at a voltage that is exactly the voltage that I have in the position of the grid if I had no grid. In this case it is like if the grid isn't existing. Now we apply the Ramo theorem.



- **A shorter SER pulse can be obtained by inserting a metal wire grid in front of the anode**
- The basic idea is that the grid acts as electrostatic screen that does not allow an electron traveling from *x=0* (cathode) to $x=w_g$ (grid) to induce charge on the anode.
- The grid bias voltage is selected to minimize the perturbation to the electron motion; i.e. it is set to the potential V_G corresponding to $x=w_g$ in absence of the grid (or slightly below it).
- In these conditions, the electric field is practically the same as in the phototube structure without grid and the motion of an electron in vacuum is also the same.

Schockley-Ramo theorem application

The electron has the same speed and trajectory of the previous vacuum tube simply because the electrode that is going out from the cathode sees exactly the same electric field. But we have a different evolution in time of the induced charge, because up until the grid, the electron is not able to induce a charge on the anode. If we apply the Ramo theorem, the grid electrode must also be grounded.



- Same electron motion as in the phototube without grid
- Different evolution in time of the induced charge on the anode.
- In fact, the reference field $\boldsymbol{E_v}$ is now very different and neatly shows that charge is induced on the anode only during the last part of the electron trajectory, i.e. from $x=w_g$ (grid) to $x=w_a$ (anode)

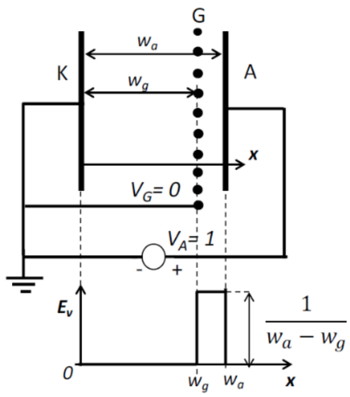
$$\begin{cases} E_v = 0 & for \quad 0 < x < w_g \\ E_v = \dfrac{1}{w_a - w_g} & for \quad w_g < x < w_a \end{cases}$$

- The SR theorem states that the SER current is

$$i_c = qE_v v_c$$

Comparison

With the grid, the single electron response is much shorter, and also higher. So we increase the BW and also SNR because the current is higher.

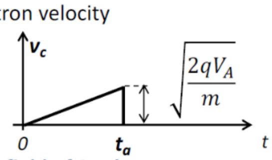


$$t_a - t_g = \sqrt{2\frac{m}{q}}\left(\frac{w_a}{\sqrt{V_A}} - \frac{w_g}{\sqrt{V_A \frac{w_g}{w_a}}}\right)$$

$$= t_a\left(1 - \frac{\sqrt{w_g}}{\sqrt{w_a}}\right)$$

True electron velocity

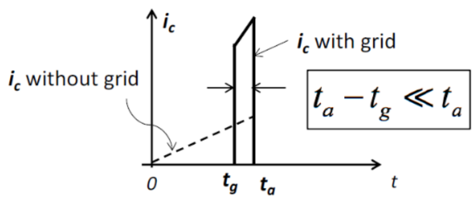
$$v_c = \frac{qV_A}{mw_a}t$$



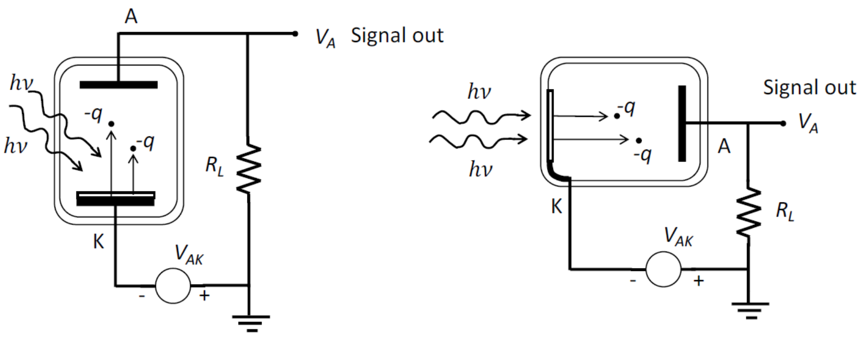
Reference field of SR theorem

$$\begin{cases} E_v = 0 & for \quad 0 < x < w_g \\ E_v = \dfrac{1}{w_a - w_g} & for \quad w_g < x < w_a \end{cases}$$

SR theorem $\quad i_c = qE_v v_c$



PHOTOTUBE DEVICE STRUCTURE



SIDE-WINDOW TUBE

- Photocathode: thick opaque layer deposited on metal support electrode
- Side window of the glass tube: unfavourable geometry, collection of light on the photocathode is uneasy and not very efficient

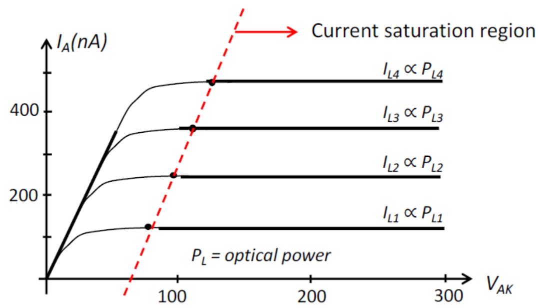
END-WINDOW TUBE

- Photocathode: thin semitransparent layer deposited on the interior of the glass tube end
- End window of the glass tube: favourable geometry, collection of light on the photocathode is easy and efficient

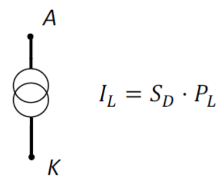
Typically is not used as in the left, because the problem is in collecting light. Since the focusing is made with lenses, focusing it on the side is very difficult. The solution is the one on the right, the back illuminated cathode.

Stationary I-V curve

- At low voltage V_{AK} the photocurrent collected at the anode is limited by the electron space charge effect
- As V_{AK} is increased the higher electric field reduces the space charge and the current increases
- As V_{AK} exceeds a saturation value V_{AKS} all photoelectrons are collected and the current is constant vs. V_{AK}
- The saturation value V_{AKS} increases with the optical power P_L on the detector
- Phototubes are operated biased into the current saturation region



PT stationary equivalent circuit: photo-controlled current generator



We have the plot of the photocurrent as a function of the bias voltage. The curve is flat, so we don't need to increase a lot the bias voltage. However, at very low bias voltage we cannot reach a steady state for the signal.
When electrons are near the anode, they can shield the electric field driving them, so the new emitted electrons see a smaller electric field → space charge effect. To overcome this issue, the bias voltage can be increased, so we need at least 100V of biasing.
In the end we will use the sensor only on the flat portion of the characteristic, so it will behave as a current generator.

PHOTOTUBE DYNAMIC RESPONSE

Main causes that limit the dynamic response:

1. Transduction from light flux to detector current: the **SER** waveform $h_D(t)$ has finite-width T_D
2. **Load circuit:** it has a low-pass filter action, δ-response $h_L(t)$ with finite-width T_L

The δ-response from light power P_L to V_A has overall shape $h_P(t)$ resulting from the cascade

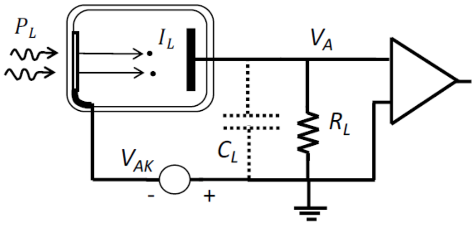
$$h_P(t) = h_D(t) * h_L(t)$$

the width T_P thus results from quadratic addition

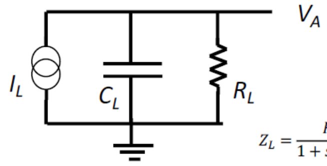
$$T_P = \sqrt{T_D^2 + T_L^2} = \sqrt{T_D^2 + R_L^2 C_L^2}$$

and for well exploiting the fast intrinsic response $h_D(t)$ of a detector it is sufficient to have

$$\boxed{T_L = R_L C_L \leq T_D}$$



PT equivalent circuit



$$Z_L = \frac{R_L}{1 + sR_L C_L}$$

Load-circuit δ-response $R_L \cdot h_L(t)$ with

$$h_L(t) = 1(t)\frac{1}{R_L C_L}\exp\left(-\frac{t}{R_L C_L}\right)$$

We have an intrinsic SER of the detector time response of Td. Then we have a load circuit, which typically includes also a capacitance, so an RC network. Are we limited by the sensor, RC or amplifier? The PT equivalent circuit is an RC network.
The SER is for us a filter intrinsic in the detector, so it is the delta response of the PT. if we have more than one filter we have to convolve the output of the detector and the delta response of the filter. We are interested in the width of the response.

At first approximation, the output time can be obtained as the square sum of the two times of the two filters. Of course we are neglecting the amplifier.
We can be either dominated by the sensor or the RC, or they can be of the same order. The best scenario is being dominated by the detector, since once we choose the detector we cannot modify it, so we are interested in optimizing all the other parts, so the electronics.

Fast response and wide active area

The light-to-current transduction by a phototube can be fairly fast, with SER pulse duration T_D around 1ns. For exploiting it, the load filtering must be adequately limited

$$\boxed{R_L C_L \leq T_D}$$

- for wide-band response low-value R_L is employed; typically, $R_L = 50\ \Omega$ to match a coaxial cable connection. With $T_D \approx$ 1ns and $R_L = 50\ \Omega$, the above requirement implies $\boxed{C_L \leq 20pF}$
- The load capacitance C_L is sum of
  C_A input capacitance of amplifier (or other circuit) connected; it can be <1pF
  C_S stray capacitance of connections; it can be < 2pF
  C_D electrode capacitance; it depends on the area A_D of the photocathode
- C_D is small even for wide sensitive area A_D , because the dielectric is vacuum and the electrode spacing is wide. In plane geometry with cathode-to-anode spacing w_a

  $$C_D = \varepsilon_0 \frac{A_D}{w_a} \qquad (\varepsilon_0 = 8{,}86\frac{pF}{m})$$

  e.g. with $w_a \approx 1cm$ it is $C_D[pF] \approx 0{,}09 A_D[cm^2]$. It's only $9pF$ for A_D=100 cm^2

- **In conclusion:** a definite advantage of Vacuum Phototubes is that they offer **very wide sensitive area together with fast response**. We will see that with semiconductor photodiodes this is not achievable

We want to get an RC value lower than 1ns, since it is the time response of the PT. Since we are interested in high speed, it is not suggested to use the optimum filter, typically fast and low noise are not compatible. Hence normally the R is 50 Ohm, the impedance of the transmission line. If the resistance is fixed, the capacitance has to be lower than 20 pF according to a Td of 1ns.

Contributions on the capacitance

- Input capacitance.
- Stray capacitance of the connections, because we are going 'outside the IC package'.
- Electrode capacitance, proportional to the area.

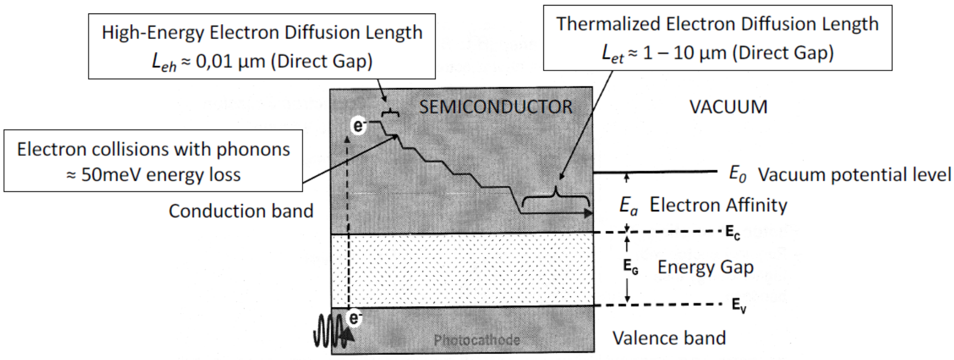
As far as the dimensions of the detector are increased, collection of light is easier, but the speed of the detector is not influenced on the area of the detector. This is a great advantage because focusing light on a detector is very difficult, e.g. if we are collecting light from a star.

ELECTRON PHOTOEMISSION AND PHOTOCATHODE TECHNOLOGY

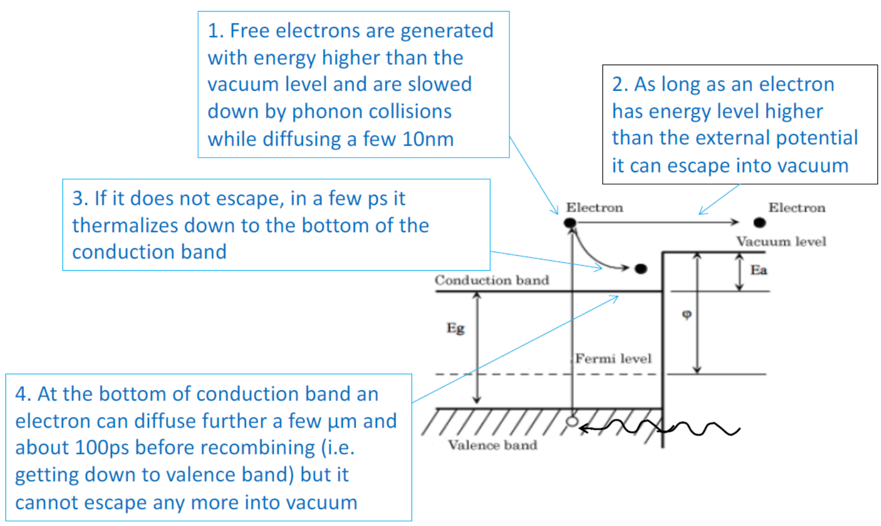
It is a three-step process:

- free electron generation by photon absorption
- electron diffusion in the photocathode layer
- escape of electron into the vacuum

Suitable materials are semiconductors. Metals are unsuitable because of the high reflectivity, small diffusion length and low escape probability (high potential step from inside up to the vacuum level).



Having an energy higher than the vacuum level is not enough to create a free electron emitted, because it has to reach the outside of the detector. So it has to have an energy higher than the vacuum level at the interface with vacuum to be released.



When the e- reaches the surface, it can either escape if the energy is higher than the vacuum level or it is lower and reabsorbed. 10nm is the distance the e- can travel without loosing too much energy.

One problem is that electrons diffuse inside the material and loose some energy before being eventually emitted. If all the energy is lost, it cannot escape. Hence to optimize this emission, the absorption length must be in the order of magnitude of the length the e- travel before loosing all the energy.
So we have to create very small detectors. As soon as we increase the absorption length, we would like also to increase the length with which the e- are travelling, but this is not possible, and in conclusion at long wavelength this type of detectors cannot collect anything.

In order to offer good quantum detection efficiency, the photocathode material must fulfill some basic requirements.

- The inside-to-vacuum energy barrier $E_g + E_a$ must be smaller than the photon energy E_p . In the visible range $1{,}6\ eV < E_p < 3{,}1\ eV$ and $E_g \approx 1eV$ for semiconductors; therefore, the electron affinity must be limited

$$E_a \leq 1eV$$

- Electrons generated in deep layers are not emitted; escape probability is high only for electrons generated in a surface layer that is very thin, about a diffusion length L_{eh} of high-energy electrons. For a significant absorption in this layer the optical penetration length L_a must anyway be NOT much higher than L_{eh} ; for a high absorption it should be comparable

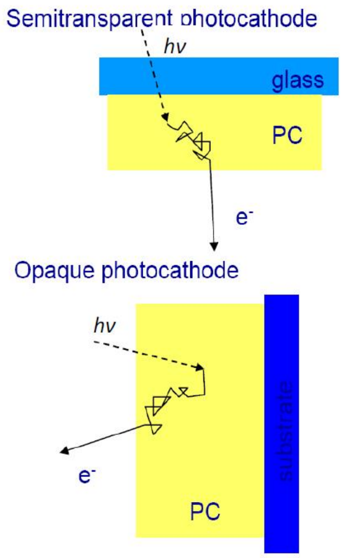
$$L_a \approx L_{eh}$$

In conclusion, the thickness of the photocathode layer contributing to the electron emission is intrinsically limited to about L_{eh} in any case. That is, the **active layer is very thin,** independent from the total thickness of the photocathode.

Semitransparent PT

The active layer of the photocathode is always very thin, also for thick cathodes deposited on a metal electrode.

This remark led to develop thin photocathodes (with thickness about $\approx L_{eh}$) deposited on the interior of the glass tube in the end-window of the detector. They are called **semitransparent cathodes.** They are illuminated on the outer side through the glass window end emit photoelectrons from the inner side. They make possible and easy a much better optical collection than the side-window geometry



Types of PT

Classifications of Photocathode types are made by *industrial standard committees*. Most widely used is that by JEDEC (Joint Electron Devices Engineering Council US), which denotes cathode types S1, S2, ... and classifies them by spectral responsivity type (rather than by chemical composition or fabrication recipe).

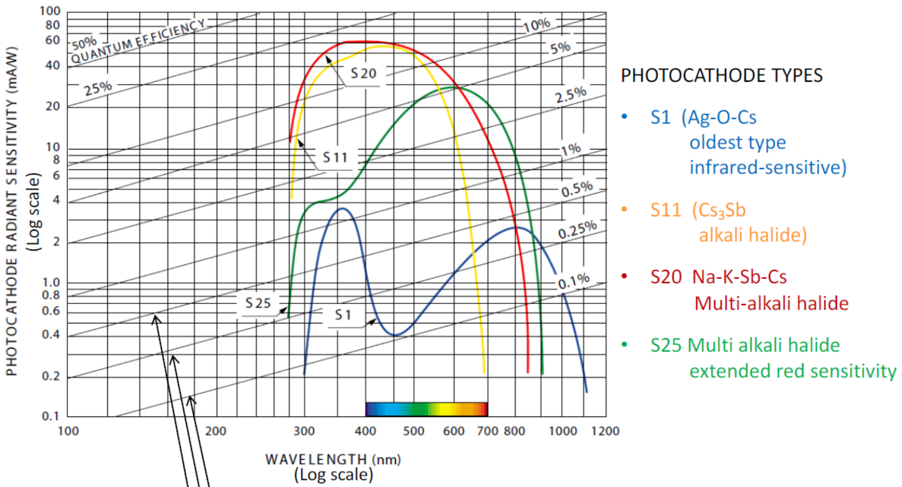
- **S1** was introduced in the '30s and is still in use. The QE is low (peak $\eta_D \approx$ 0,4% at =800nm) but covers a wide spectrum in the IR. It is a matrix of Cesium oxide that includes silver microparticles and it's currently denoted Ag-O-Cs.

Highly efficient photocathodes for the visible range were introduced in the '50s and progressively developed employing compounds of alkali metals (Na, K, Cs, which have low work functions) and Antimony (Sb). Main types:

- **S11** ranges from 300nm to 600nm, peak $\eta_D \approx$15% at 450nm; alkali halide Cs_3Sb
- **S20** ranges from 300nm to 800nm, peak $\eta_D \approx$20% at 350nm; multi-alkali halide Na-K-Sb-Cs
- **S25** extends the range up to 800nm, peak $\eta_D \approx$5% at 600nm; multi-alkali Na-K-Sb-Cs like S20, but with a thicker layer that gives higher sensitivity in the red, at the cost of lower sensitivity in the blue-green

Quantum efficiency is 0.4% at 800nm, so not particularly efficient. S11 cannot instead detect at 800nm. If we increase the efficiency we are shifting the wavelength range.

Radiant Sensitivity or Spectral responsivity



NB: the auxiliary lines marked with Quantum Detection Efficiency (QE) in %
make possible to read directly from the diagram also the QE

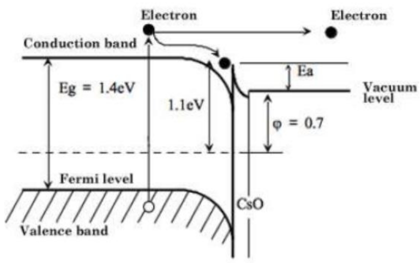
The lines are the lines of fixed quantum detector efficiency, and we notice that there is no cathode with a quantum detector efficiency higher than 25%. So **we cannot use them at a wavelength higher than 800nm**.

PT WITH NEGATIVE Ea

We want to overcome the problem of e- loosing energy before reaching the surface and hence not being able to escape. We use few atomic layers of Cs-O to vent the band diagram to have an electron energy that is higher than the vacuum level, so we are able to escape outside with an electron affinity lower than the conduction band. So the quantum efficiency is higher also at long wavelength, the drawback is that noise is increased. However, lambda is still 900nm, not um.

Progress in semiconductor physics and technology led in the ’70s to devise a new class of photocathodes, called photocathodes with Negative Electron Affinity (NEA)

- On a GaAs crystal substrate, a **few atomic layers of Cesium Oxide (*Cs-O*) are deposited and activated**, thus forming a very thin positive charge layer of Cs^+ ions.
- **The electric field** generated at the surface **curves downward the energy bands**: the vacuum potential level is now lower than the bottom of conduction band, i.e. the electron affinity E_a is negative
- Electrons can now escape into vacuum also when thermalized at the bottom of conduction band; QE is thus enhanced
- Photoelectron emission is obtained also with photons with lower energy E_p , down to the GaAs energy gap E_g



In conclusion: NEA cathodes offer **higher QE** value and **broader spectral range**, extending up to the absorption edge of GaAs (i.e. $\lambda \approx 900nm$ corresponding to the gap $E_g \approx 1,4\ eV$)

DARK CURRENT AND NOISE

- A finite current is emitted by any photocathode even when kept in the dark, without any light falling on it.
- It is a spontaneous emission due to thermal effects (phonon-electron interactions in the cathode) and is called **Dark Current**.
- The dark current density j_B (per unit area of cathode) depends on the cathode type and on the cathode temperature. Typical values at room temperature are reported in the Table

PhotoCathode type	Dark Current density j_B in A/cm^2	Dark Electron Rate density n_B in electrons/s·cm^2
S1	$\approx 10^{-13}$	$\approx 10^{6}$
S11	10^{-16} - 10^{-15}	10^{3} - 10^{4}
S20 and S25	10^{-19} - 10^{-16}	1 - 10^{3}
GaAs NEA	10^{-18} - 10^{-16}	10 - 10^{3}

Electrons can also be generated with no light, that is the dark current. This is noise, electrons thermally generated, and we cannot distinguish a thermal or optical generated photon, so if it is noise or signal.

Thermal generation is in the order of 10^3 electrons per square cm (neglecting S1 case), which is quite small with respect to the optical generated ones.
The problem of dark current is not a problem itself, the fact that it gives us an offset is never a problem, we measure is and subtract it. The problem is not the absolute value of dark current, but the shot noise associated to it; in fact, we can remove the dark current but not the shot noise associated to it.

Detector internal noise

Once we have the thermally generated e- we can compute the current, which is shot noise.
This value is not important for <u>this kind of detectors because this noise is negligible compared to any other source of noise</u>.

The total Dark Current is $I_B = j_B A_D$ where A_D is the area of the photocathode.

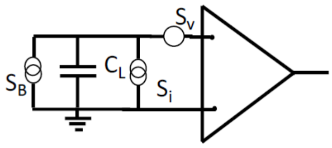
The shot noise of I_B is the photodetector unavoidable internal noise, with effective power density (unilateral)

$$\sqrt{S_B} = \sqrt{2qI_B} = \sqrt{2qj_B}\sqrt{A_D}$$

Typical values of $\sqrt{S_B}$ are reported in the Table

PhotoCathode type	Dark Current density j_B A/cm^2	Shot Noise Effective density $\sqrt{S_B}$ $pA/\sqrt{Hz}\sqrt{cm^2}$
S1	$\approx 10^{-13}$	$\approx 10^{-4}$
S11	10^{-16} - 10^{-15}	$\approx 10^{-5}$
S20 and S25	10^{-19} - 10^{-16}	$\approx 10^{-7} - 10^{-6}$
GaAs NEA	10^{-18} - 10^{-16}	$\approx 10^{-6}$

Amplifier's noise



- We know that for operating with low-noise a high impedance sensor must be connected to a preamplifier with high input impedance and low input noise. The best available preamplifiers have current noise at room temperature

$$\sqrt{S_i} \approx 0{,}01 \frac{pA}{\sqrt{Hz}}$$

- **The circuit noise $\sqrt{S_i}$ is always dominant** and the detector **internal noise $\sqrt{S_B}$ plays in practice no role with any phototube**, even for detectors with S1 photocathodes (that have the highest noise) and even with very wide sensitive area (up to many square centimeters). In fact, for producing shot noise with power density higher than that of the circuit noise, the phototube dark current should be I_B>300pA, corresponding to an emission rate $n_B > 10^9$ electrons/s.
- **Vacuum tube photodiodes can thus be employed for operating at low noise without stringent limits to the sensitive area**. As we will see, this is a definite advantage over semiconductor photodiodes.

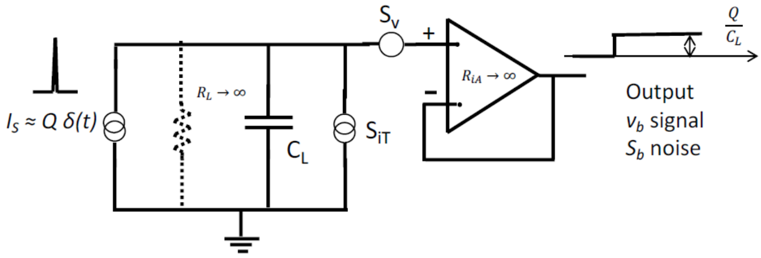
Current noise of the amplifier is dominant over the one of the PT. with the PT we don't create large device because they are unuseful, not because the noise is increasing, or better, is always negligible with respect to the noise of the amplifier.

LOW NOISE PREAMPLIFIERS FOR PHOTODIODES

Let's put aside the speed now to focus on the best possible SNR.

Voltage buffer preamplifier

- Photodiodes are high-impedance sensors (both the vacuum phototubes and the semiconductor photodiodes), hence for low-noise operation they must be connected to preamplifiers with high input resistance* $R_{iA} \rightarrow \infty$ (see slides in OPF2)
- Simple configuration: voltage buffer based on a high-input-impedance and low-noise amplifier



- C_L total load capacitance = C_D (detector cap.) + C_{iA} (amplifier cap.) + C_S (connection cap.)
- R_L total load resistance → ∞
- S_v amplifier voltage noise
- S_{iT} total current noise = S_{iD} detector noise + S_{iA} amplifier noise (+ S_{iR} load resistor noise)

* R_{iA} = true physical resistance between the input terminals, not the dynamic input resistance including feedback effects

Buffer voltage output:

Step signal	$v_b(t) = \frac{Q}{C_L} \cdot 1(t)$
Noise Spectrum	$S_b = S_v + S_{iT} \frac{1}{\omega^2 C_L^2}$

The buffer configuration has some noteworthy drawbacks.

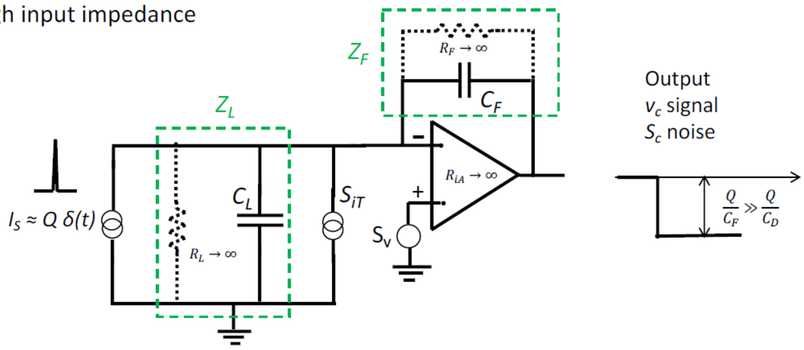
- The signal amplitude Q/C_L is ruled by the total capacitance $C_L = C_D + C_{iA} + C_s$, whose value is not very small and not well controllable, particularly in cases where long sensor-preamplifier connections contribute a remarkable C_s.
  C_L may be different from sample to sample of the amplifier, even of the same amplifier model.
- With signals in high-rate sequence, the superposition of voltage steps may build-up and produce a significant decrease of the photodiode bias voltage. This may change the operating conditions and consequently the parameters and performance of the detector, particularly if the photodiode is biased not much above the saturation voltage.

The problem is the capacitance C_L, which is the capacitance of the detector Cd, small, the one of the amplifier Ca and of the connection Cs. So we might be in situations where the Ca and Cs are dominating. The idea is to change this approach (for semiconductor devices this problem is smaller since we have smaller area and so smaller capacitance).
Furthermore, with large pulses in input we change the voltage across C_L, so we are also changing the bias voltage of the PT.

The solution is the following.

Charge Preamplifier

Alternative configuration: operational integrator based on a low-noise amplifier with high input impedance



- C_F capacitor in feedback. The **C_F value can be very small and is accurately set** by the capacitor component, because the inherent stray capacitance between output and input pins of the amplifier is negligible. Therefore, **one can work with $C_F \ll C_L$**
- R_F feedback resistor → ∞
- C_L total load capacitance = C_D (detector cap.) + C_{iA} (amplifier cap.) + C_S (connection cap.)
- R_L total load resistance → ∞
- S_v amplifier voltage noise
- S_{iT} total current noise = S_{iD} detector noise + S_{iA} amplifier noise (+ S_{iR} load resistor noise)

We have our sensor and the stray capacitance C_L, but we use a transimpedance amplifier and we put Cf in feedback.
We can work with Cf much lower than C_L, so with a signal higher than the previous case. If we look at the signal, the effect of Cf is that we have Q/Cf instead of Q/C_L.

Output Signal:

in frequency domain $V_c = -QZ_F = -\dfrac{Q}{j\omega C_F}$ in time $v_c(t) = -\dfrac{Q}{C_F}\cdot 1(t)$

With respect to the buffer, the amplitude is greater by the gain factor $G_c = C_L/C_F \gg 1$

$$\boxed{|v_c| = \frac{Q}{C_F} = \frac{C_L}{C_F}\cdot\frac{Q}{C_L} = \frac{C_L}{C_F}\cdot|v_b| = G_c\cdot|v_b|}$$

Advantages:

- The higher signal makes less relevant the noise of the following circuits
- The signal amplitude is ruled by the well controlled and stable C_F, no more by the other capacitances C_D, C_{iA} and C_s
- The detector terminal is connected to the amplifier virtual ground, hence it stays at constant bias voltage even with signals in high-rate sequence

The noise analysis (see next slide) confirms that these advantages are obtained without degrading the S/N. The charge amplifier configuration thus is the solution of choice in most cases met in practice.

We define a gain C_L/Cf just to compare the situation with the previous one.

Output Noise Spectrum :

- the current noise S_{iT} is processed by the same transfer function as the current signal
- the voltage noise S_v is processed with the transfer function from non-inverting input to amplifier output.

Denoting by Z_L the load impedance and by Z_F the feedback impedance

$$S_c = S_v \left|1 + \frac{Z_F}{Z_L}\right|^2 + S_{iT}|Z_F|^2$$

in our case $Z_L \approx 1/j\omega C_L$ and $Z_F \approx 1/j\omega C_F$ so that

$$S_c = S_v \left|1 + \frac{C_L}{C_F}\right|^2 + S_{iT}\frac{1}{\omega^2 C_F^2} = \left(\frac{C_L}{C_F}\right)^2 \left[S_v\left(1 + \frac{C_F}{C_L}\right)^2 + S_{iT}\frac{1}{\omega^2 C_L^2}\right]$$

if $C_F/C_L << 1$, with good approximation it is

X $$\boxed{S_c \approx \left(\frac{C_L}{C_F}\right)^2 \left[S_v + S_{iT}\frac{1}{\omega^2 C_L^2}\right] = \left(\frac{C_L}{C_F}\right)^2 S_b = G_c^2 S_b}$$

With respect to the buffer, the signal and noise thus benefit of the same gain G_c : therefore, the attainable S/N is the same with the charge preamplifier as with the voltage buffer preamplifier

X is the WN of before times the gain squared. So we are getting the same SNR but with some advantages. We are not just adding a gain, in fact we are changing the configuration when saying Cf << C_L.

NEP AND DETECTIVITY

- Evaluations and comparisons of Photocathodes are currently based on the **Noise Equivalent Power NEP**, a figure of merit that takes into account the photon detection efficiency and the detector dark-current noise, but not the preamplifier noise.
- NEP is defined with reference to a situation where **the limit** to the minimum measurable signal is **set by the internal noise of the detector** and not by the electronic circuit noise. We have seen that this is **NOT the case with PhotoTubes** but we will see that **it is the case with PhotoMultiplier Tubes**. NEP was devised as an figure of merit for comparing objectively the intrinsic quality of different detectors.

Let a photocathode have area A_D , signal current I_p and Dark Current I_B with area density j_B . Employing a filter with bandwidth (unilateral) Δf we have noise

$$\sqrt{\overline{i_n^2}} = \sqrt{2qI_B\Delta f} = \sqrt{2qj_B}\sqrt{A_D}\sqrt{\Delta f} \qquad \text{and} \qquad \frac{S}{N} = \frac{I_p}{\sqrt{\overline{i_n^2}}}$$

The minimum measurable current signal $I_{p,min}$ (corresponding to S/N=1) is

$$I_{p,\,min} = \sqrt{\overline{i_n^2}} = \sqrt{2qj_B}\sqrt{A_D}\sqrt{\Delta f}$$

For illumination with optical power P_p at a given λ the Detector Responsivity is

$$S_D = \frac{I_p}{P_p} = \eta_D \cdot \frac{\lambda}{\frac{hc}{q}} = \eta_D \cdot \frac{\lambda[\mu m]}{1{,}24}$$

NEP is the Noise Equivalent Power, which is unuseful for the PT. It is the minimum optical power I can detect to have a SNR = 1 if the only noise present is the noise of the detector. It is nosense on the PT because I'm never limited by the noise of the detector, but by the electronics typically. In a lot of other detectors this is not the case, however.
NEP is minimum current divided by radiant sensitivity. It is an expression that works if we are limited by the noise of the detector.

NEP depends on the bandwidth and on the area. However, the bandwidth is connected to the application we are using the detector in, not strictly to the detector. Moreover, also the area of the detector should be a choice of the designer depending on the application.

So a new figure of merit is introduced, the detectivity. It is the bandwidth and area divided by the NEP.

- NEP is defined as the input optical power $P_{p,\,min}$ corresponding to the minimum measurable signal

$$NEP = P_{p,\min} = \frac{I_{p,\min}}{S_D} = \frac{\sqrt{\overline{i_n^2}}}{S_D} = \frac{\sqrt{2qj_B}\sqrt{A_D}\sqrt{\Delta f}}{S_D}$$

  In essence: NEP = detector noise referred to the input (in this case the **optical input**).

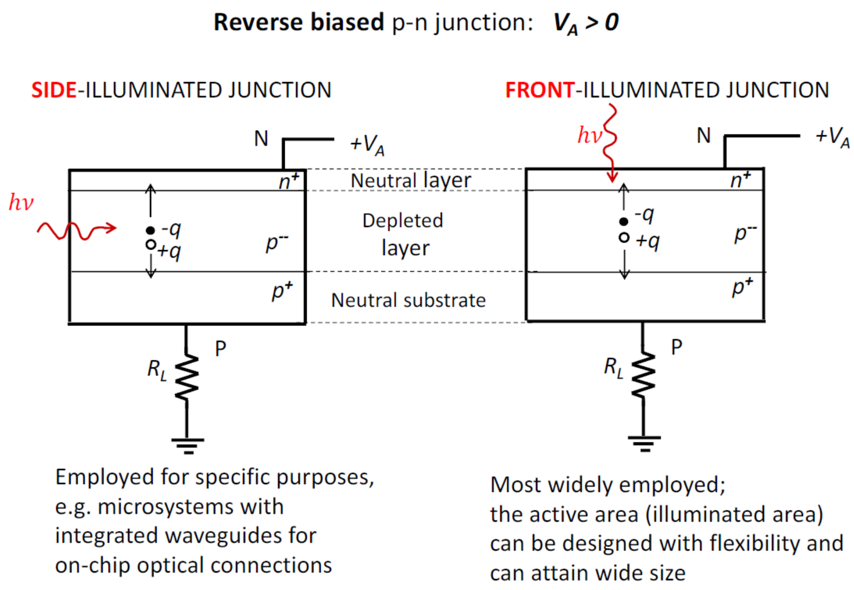
- However, the NEP is not a fully objective figure of merit for assessing and comparing the quality of photocathodes: in fact, **cathodes of equal quality have different NEP if they have different area**. Furthermore, the NEP is an inverse scale, that is, the best photocathodes have the lowest NEP figures.
- A different figure named Detectivity D* was therefore derived from the NEP by
  a) considering the NEP value normalized to unit sensitive area ($A_D = 1cm^2$) and to unit filtering bandwidth (Δf = 1Hz)
  b) defining the Detectivity D* as the reciprocal of the normalized NEP

$$\boxed{D^* = \frac{\sqrt{A_D}\sqrt{\Delta f}}{NEP}} \quad \text{that is} \quad D^* = \frac{S_D}{\sqrt{2qj_B}} = \eta_D \cdot \frac{\lambda[\mu m]}{1{,}24} \frac{1}{\sqrt{2qj_B}}$$

The detectivity describes how much good is the detector we are developing.

PHOTODIODE DEVICES

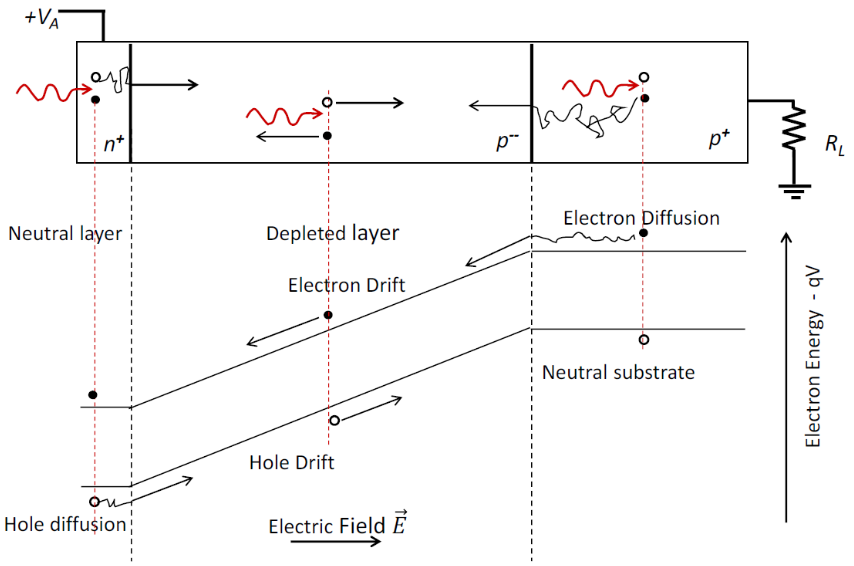
CARRIER MOTION



Same situation we have in the PT, and also here we can have a side illumination, even If the front illuminated junction can be used. Side illuminated device has a problem that can also be a good thing; normally, the diameter of these device is in the order of 50/100 um, much smaller than the photocathode. The problem is that the thickness of the depleted region is in the order of 1um, and focusing the light here is a problem. So the 'height' is 1 um and the length 50 um, and all this length can be used to absorb the light, even if the entrance light dimension is small. So I can absorb all the light with 50 um. In the front-side illuminated I cannot absorb all the light.

We can however focus light with a mono-mode fiber. However, the real advantage of this side-illuminated structure is when I don't have to focus the light in.

Carrier motion in PD



Differently from the pn junction, in the PD we have to consider also upper neutral layer and substrate. If light comes from the left to the right, it seems that it is absorbed in the depleted region, but it could be absorbed also in the neutral layer.
In the neutral layer or substrate we are generating still a carrier, but it is not travelling because there is no electric field, so it moves around, but it is also surrounded, in the neutral layer, by a lot of other carriers, so the anode cannot see it because it is like 'shielded'. Hence **in the neutral substrate and neutral layer the carrier is not generating a current**.

However, if the carrier is travelling in the neutral substrate and after a while with a random motion it reaches the depleted region, it is no more shielded, it sees the electric field and it generates a current. The problem is that there is a strong delay between when the carrier is generated is and the current is.

Carriers generated in the depleted layer:

- A carrier in the depleted layer induces opposite charges in the conductive electrodes (neutral semiconductor layer and metal contact to the external circuit)
- The value of the induced charge on a given electrode depends on the carrier distance from the electrode
- If the carrier moves the **charge induced on the electrode varies**, hence current flows through the contact

Conclusion: a carrier drifting in the depleted layer **causes current to flow** through the metal contact to the external circuit

Carriers generated in neutral regions:

- A carrier in a neutral region is surrounded by a huge population of other free carriers
- When the carrier moves the distribution of free carriers swifltly rearranges itself to electrically **screen any effect of the carrier motion** on the external circuit

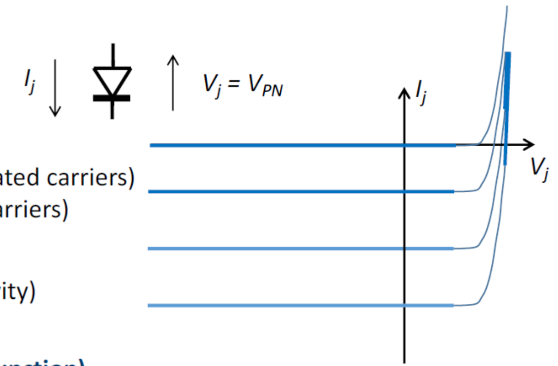
Conclusion: **as long as it diffuses** in a neutral region, a carrier **does NOT cause current** to flow through the metal contact to the external circuit.
However, **if** by diffusion it reaches the edge of depletion layer before recombining, **then** it drifts in the electric field and causes current to flow.

We want to calculate the signal.

I-V characteristic of PD

❑ I-V of illuminated p-n junction

$$\boxed{I_j = I_0 \exp\frac{qV_j}{kT} - I_0 - I_p}$$



I_0 reverse current (thermally generated carriers)
I_p photocurrent (photogenerated carriers)

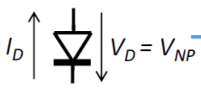
$$\boxed{I_p = S_D \cdot P_L}$$

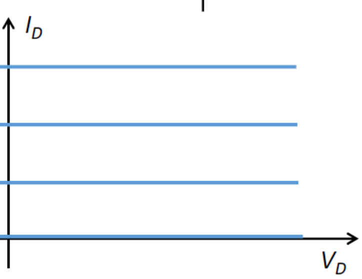
(P_L optical power; S_d radiant sensitivity)

❑ Photodetector (reverse-biased junction)

I_D detector current = - I_j
V_D detector voltage = - V_j

$$I_D = I_p + I_0 - I_0 \exp\left(-\frac{qV_D}{kT}\right)$$





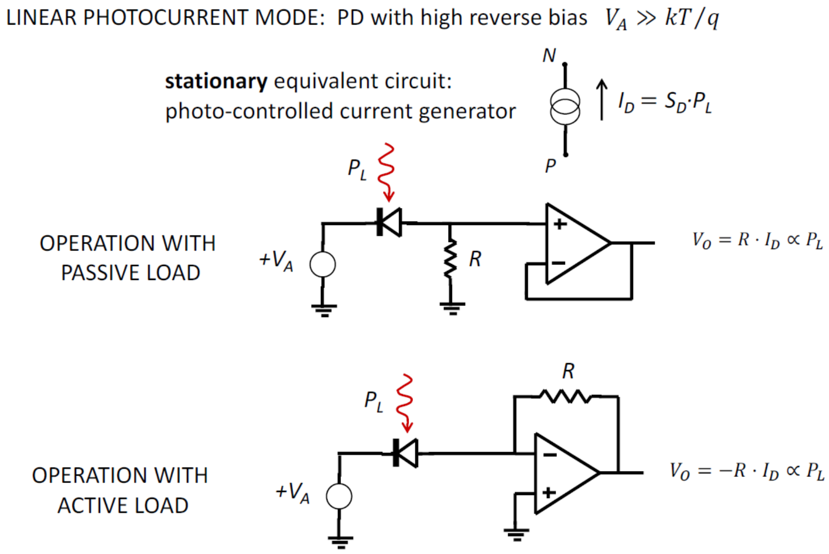
with $V_D \gg kT/q$ $\boxed{I_D = I_p + I_0 \approx I_p}$

Detector photocurrent $\propto P_L$

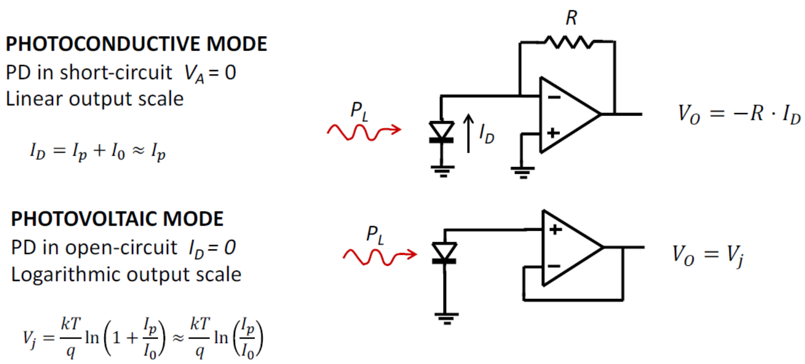
Detector dark current

It is a diode current with an offset that changes depending on the optical generated current. The PD is always used in a revere bias mode, because we need depleted region to collect light, which increases increasing the reverse voltage.

PD operation modes



Semiconductor photodiodes can be operated also without a bias voltage source. As outlined below, the short-circuit current is measured in the photoconductive mode and the open-circuit voltage in the photovoltaic mode. These configurations have modest sensitivity and slow response (see later), but their simplicity is attractive in some practical cases, e.g. for monitoring a steady light over a wide dynamic range.



PHOTON DETECTION EFFICIENCY

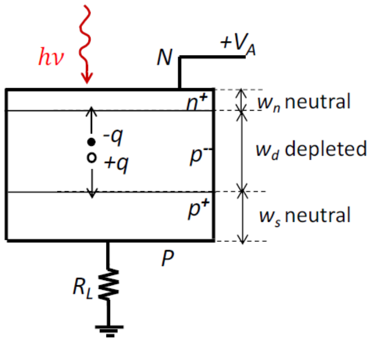
It has to take into consideration 4 different aspects (light comes from the top): number of photons absorbed in the depleted region, or in the opposite way the photons absorbed in the neutral region. The last is the number of photons reflected on the surface.

P_d = probability of a photon to generate a free electron-hole pair **in the depletion layer** = product of probabilities of

1. NOT being reflected at the surface
2. NOT being absorbed in the top neutral layer w_n
3. BEING absorbed in the depletion layer w_d

Denoting by R the reflectivity (probability of reflection) and L_a=$1/\alpha$ optical absorption depth:

$$P_d = (1-R)\cdot e^{-\alpha w_n}\cdot(1-e^{-\alpha w_d})$$



In most PD structures the probability that carriers photogenerated in neutral regions reach by diffusion the depletion layer is negligible, hence the photon detection efficiency or quantum detection efficiency η_D is simply

$$\boxed{\eta_D \;=\; P_d \;=\; (1-R)\cdot e^{-\frac{w_n}{L_a}}\cdot\left(1-e^{-\frac{w_d}{L_a}}\right)}$$

In PD structures where carriers diffusing in neutral regions have significant probability of reaching the depletion region, additional contributions to η_D must be taken into account

1-R is the amount of light that is not reflected. Than we have the exponential decay time of the light that is absorbed in the neutral region, which is lost at this moment. Then we have 1 - light absorbed in the depleted region, which is the light that escapes from the bottom.
From this formula we can compute the detection efficiency, which is the percentage of light we can use to create a signal.

In the PD we can choose any parameter in the formula in the square box, not as a designer, but as a user, looking at the catalogues. Only the thickness of the neutral region (absorption length) is not of choice because the manufacturer tries to reduce it as much as possible.
To increase the dimension of the depleted region, we can increase the applied voltage, and sometimes it is an issue, also because we have the problem of power dissipation. So reducing the bias voltage seems good. However, I also want a big depletion region because I want a high detection efficiency.

Eta_D

$$\boxed{\eta_D \;=\; P_d \;=\; (1-R)\cdot e^{-\frac{w_n}{L_a}}\cdot\left(1-e^{-\frac{w_d}{L_a}}\right)}$$

Basic sources of η_D losses are 1) surface reflection, 2) absorption in the neutral input layer and 3) incomplete absorption in the depletion layer (active volume).
The η_D value attained depends on the actual material properties and PD structure and on the light wavelength λ.

η_D loss by Reflection

- The **reflection at vacuum-semiconductor surface is strong** because of the high step discontinuity in refractive index n, since n is high in semiconductors. In Silicon n>3,5 over all the visible range and further rises at short λ; the reflectivity is accordingly high R>30% and further rises at short λ .
- Losses can be reduced by **tapering the n-transition** with deposition of a multi-layer anti-reflection (AR) coating of materials with n values suitably scaled down from semiconductor to vacuum. Strong reduction can be obtained, down to R<<10%.
- In Silicon PDs a **simple AR coating** is obtained with a surface oxide layer (passivation layer), because SiO_2 has intermediate n≈2. Remarkable reduction can be obtained, down to R≈10%.

Reflective index of Silicon is quite high and we can use an antireflection coating, and also the reflective index of SiO2 is in the middle between air and silicon, so putting a layer of it can reduce the reflectivity below 10%.

As for the losses in the cathode neutral region, this is a problem because at this moment, at the first order, all the light absorbed in the upper neutral region is lost. So we should reduce it, but it is difficult to make a very thin upper neutral region form a technological standpoint.
For the depleted region, the problem is the tradeoff between power dissipation and amount of light to be collected.

$$\boxed{\eta_D \;=\; P_d \;=\; (1-R)\cdot e^{-\frac{w_n}{L_a}}\cdot\left(1-e^{-\frac{w_d}{L_a}}\right)}$$

η_D loss by absorption in neutral input layer

- At short λ, η_D cutoff occurs because photons are all absorbed in the neutral region at the surface. The escape probability is ruled by w_n/L_a (see 2nd term).
  In Silicon L_a is small at short λ : $L_a < 1\ \mu m$ for $\lambda < 500 nm$ and $L_a < 100\ nm$ for λ<400nm. In actual Si-PD structures w_n ranges from about *200 nm* to *2 μm;* the cutoff λ congruently ranges from about *300 nm* to *400 nm*.

η_D loss by incomplete absorption in the depletion layer

- At long λ , η_D cutoff occurs because the absorption falls down. Absorption is ruled by w_d/L_a (see 3d term); with $w_d/L_a << 1$ we get $\left(1-e^{-w_d/L_a}\right) \approx w_d/L_a$.
  Silicon is ≈ transparent beyond *1100 nm*, since photon energy < Si energy gap. In actual Si-PD structures the depth w_d can range from one to various tens of μm; given the λ-dependance of L_a , the cutoff λ ranges from about *900 nm* to *1100 nm*.

Current Si-PDs provide high efficiency ($\eta_D > 30\%$) in the visible 400nm < λ < 800nm.

The operation range can be extended to longer λ with PDs in other semiconductors: up to 1500nm with Germanium devices and up to 2000nm with InGaAs devices

DARK CURRENT AND NOISE

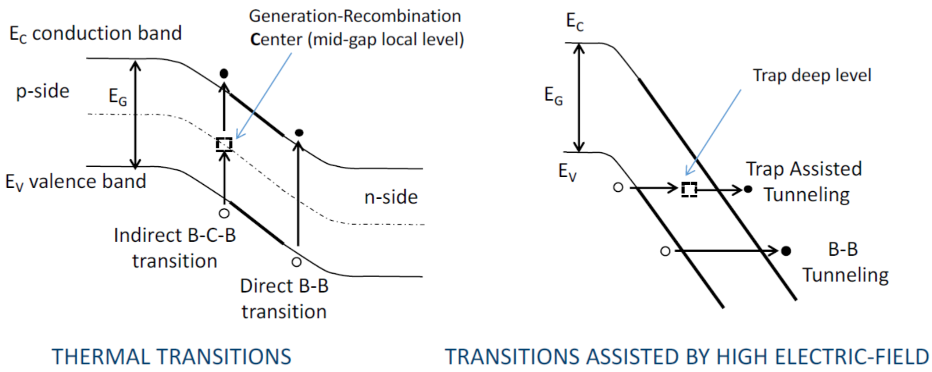
The noise for us is what we have when there is no signal, i.e. in the dark since we have light sensors. Dark current is not a problem at all, we don't have a problem with the baseline or background, the problem is the shot noise associated to the baseline, background or dark current.
Normally, dark current is due to thermally generated carriers, which create a current with a shot noise associated to it.

- Even without light falling on it, a finite current I_B flows in a reverse-biased p-n junction. It is called **Dark Current** in PDs and reverse current in ordinary circuit component diodes.
- I_B is due to spontaneous generation of free carriers by thermal effects (and also by tunnel effects in device structures with high electric field).
- Just like in Phototubes, the shot noise of I_B is the photodiode internal noise, with effective power density (unilateral)

$$\sqrt{S_B} = \sqrt{2qI_B}$$

- The internal noise of PD devices with **microelectronic-size** (sensitive area <1mm^2) is much lower than the input noise of even the best high-impedance preamplifiers. In the applications of microelectronic PDs the circuit noise is dominant, just like for vacuum phototubes.
- However, semiconductor PDs have **dark current density j_B much higher than vacuum phototubes**; this fact significantly limits the active area size of semiconductor detectors that can be employed for very low-noise operation.

With the PD the problem of G/R processes is higher than the PT. In the PD we have thermal generation as usual, but also tunneling. Tunneling increases as soon as we increase the electric field, so it might be reasonable to have a small bias voltage. However, increasing the bias voltage could be useful to increase the depleted region and to introduce a gain.



- Various physical phenomena take part in carrier generation-recombination, with varying relative relevance in the various cases, with different materials, device structures and operating conditions (bias voltage, temperature, etc.).
- Silicon has very favourable properties for achieving low generation rate.
- Materials for **IR detectors (Ge, InGaAs) have smaller energy gap and therefore inherently higher noise**, since all generation processes are favoured by a smaller E_G

Dark current of Silicon-PD

In Silicon device physics and technology it is ascertained that in reverse-biased junctions with moderate electric field intensity:

a) the dark current is mainly due to thermal generation of carriers in the depletion layer. Contribution by diffusion of minority carriers from neighbouring neutral regions are much lower and negligible in comparison.

b) The thermal generation rate in the depletion has volume density n_G given by

$$\boxed{n_G = \frac{n_i}{2\tau}}$$

n_i = intrinsic carrier density; $n_i = 1{,}45 \times 10^{10}$ cm^{-3} @ Room Temperature

τ = minority carrier lifetime, strongly dependent on the device technology i.e on the starting material and on the fabrication process. Typical values:

$\tau \approx \mu s$ ordinary Si technology for integrated circuits

$\tau \approx ms$ ordinary Si technology for detector devices

$\tau \approx 1 \div 10s$ best available Si technology for detector devices

Tau is specific of the material we are using. It defines the amount of time we need to have thermal generation, and it depends in a strong way on the material we are using and, once fixed the material, on the substrate (high quality or low quality silicon). Also with the same substrate, if we change the process to develop the device we could change the quality and the tau.

Dark current and active area of Si-PD

Of course, the thermal generation must be multiplied by the volume, which is the area times the length. So which is the maximum area, i.e. maximum diameter of the diode we can use, if we fix the noise? The result is in the formula x.

To understand this, we can make some examples. We want the widest possible area with noise lower than the preamplifier. Looking at the result, the diameter must be smaller than 1.3 cm. with 10 cm, the noise was already negligible for the PT, while with PD we have a noise comparable with the amplifier

with 1.3 cm of PD. So with the PT, the noise of the PT is negligible with respect to the amplifier, with the PD it depends on the diameter of the PD.
So let's try to compare directly the PT and the PD. We want to understand the maximum diameter of the PD to have a noise that is comparable to the one of the PT. The result is 130 um, with a standard dimension of a PT in the order of inches (2.5cm), so we are comparing um with cm, and this is not good.

A Si-PD with circular active area of diameter *D* (area $A = \pi D^2/4$) and depletion layer thickness w_d has dark generation rate $n_B = n_G A w$. For setting a limit $n_B < n_{B\max}$ the diameter *D* must be limited

$$A < A_{max} \frac{n_{B\max}}{n_G w_d} = \frac{2\tau n_{B\max}}{n_i w_d} \qquad D \le D_{max} \sqrt{\frac{8\tau n_{B\max}}{\pi n_i w_d}} \quad \text{x}$$

Example: Si-PD with w_d =10µ in good Si detector technology ($\tau \approx$10ms), intended to have the widest possible area with noise lower than a preamplifier with $\sqrt{S_i} = \approx 0{,}01 pA/\sqrt{Hz}$. For keeping the shot noise so low, the generation rate must be limited to $n_{B\max} < 10^9 s^{-1}$ which implies

$$D < D_{\max}\text{=1,3cm}$$

As we will see, the area limitation is more severe for avalanche photodiodes (APD). The APD internal gain makes negligible the role of circuit noise, hence it is the APD detector noise that limits the sensitivity and it is worth to reduce it more drastically.

Example: Si-APD with *w =10µm,* fabricated in very good Si detector technology (say $\tau \approx 1s$) intended to have low dark rate, comparable to that of a good vacuum tube photocathode, say $n_{B\max} < 10^3 s^{-1}$ like a S20 photocathode with diameter 3cm. The limit is

$$D < D_{\max} = 130um$$

CURRENT SIGNAL IN PDs

Current signal involves the Ramo theorem; the problem is that it is really difficult.

CARRIER MOTION AND DETECTOR CURRENT

With the Ramo theorem we need the speed of the carrier in the device in terms of absolute value and direction, then we have to compute the reference electric field, we multiply them and we get the result.
With the pn junction we don't have just the drift current from cathode to anode, but we have a crystal and so the motion is not linear, because the crystal vibrates. Hence computing the real trajectory of the carrier is very difficult.
Moreover, also for the absolute value we have a problem, because with the pn junction the speed of the carrier is proportional to the electric field, which is not constant in the junction. Moreover, the speed is proportional to the electric field only up to a certain level, then the relation saturates and we have velocity saturation regardless the applied electric field.

Hence making a real calculation of the Ramo theorem in a pn junction is not piece of cake. So we need to make some hypothesis.

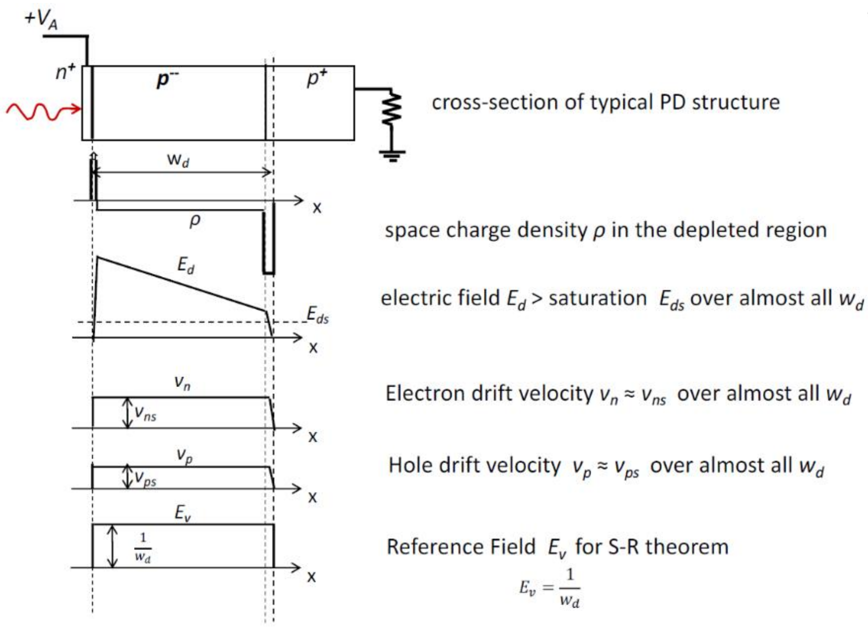
- Carriers drifting in depleted regions induce current at PD terminals, whereas carriers diffusing in neutral regions do NOT
- **The Shockley-Ramo (S-R) theorem is still valid in presence of space charge**
- Knowing the actual velocity v_c of a drifting carrier, the current induced at the PD terminals can be computed by the S-R theorem
- **The motion** of carriers in a semiconductor with electric field E_d **is different from that in vacuum with equal E_d** : carriers suffer scattering on the lattice and dissipate in the collisions most of the energy received from the field. No more the acceleration, but the drift velocity v_c is a function of the field E_d .
- In Silicon (and other materials) the motion of electrons is different from holes:
  - at **low field E_d <2 kV/cm** = *0,2 V/μm* the regime is **Ohmic**: $v_c = \mu_c E_d$
    (electron mobility $\mu_n \approx 1500\ cm^2V^{-1}s^{-1}$; holes $\mu_p \approx 450\ cm^2V$)
  - as E_d increases above *2kV/cm* the velocity rises progressively slower
  - at **$E_{ds} \approx$ 20kV/cm** = *2V/μm* the **velocity saturates** at the scattering-limited values

  for electrons $\boxed{v_{ns} \approx 10^7\ cm/s}$ for holes $\boxed{v_{ps} \approx 8 \cdot 10^6\ cm/s}$

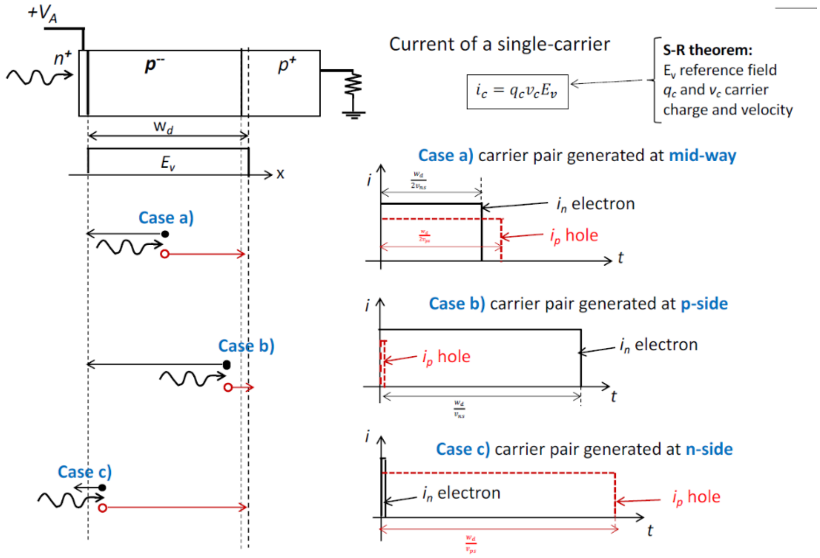
  which are almost equal to the thermal scattering velocity $v_{th} \approx 10^7\ cm/s$

Carrier motion in PD

For the reference electric field we have to remove all the carriers in the junction for the Ramo theorem, so we have a capacitor. The problem is that the real electric field that gives the speed of the carrier is not constant, and moreover the electric field is always bigger than the electric field that causes velocity saturation. In fact, if it is high, we maximize the speed of the carriers and so the response of the device. If Ed is so high everywhere that the speed is saturated, the velocity of hole and e- is saturated, even if the trajectory is not linear.



However, there is another problem, that is the fact that we don't know where light is absorbed the carrier in the depleted region. So we have a carrier due to e- and one due to the hole. If the photon is absorbed close to the border, the hole reaches immediately the contact, so the current of the hole disappears immediately, while the one due to e- lasts more. We can also have the opposite case, where I have only the current due to the hole.



if we have the single electron response, if we have a signal that is more complicated, I can convolve the single electron response with the number of photons we get to get the signal shape. The problem with the PD is that as soon as we change the absorption point, we change the single electron response. We know that even if we consider deterministic systems, the absorption in the depleted region is exponential, but it is an average absorption, it doesn't mean that every photon is absorbed at that thickness. It is a statistical process.

Taking into account all these things makes impossible a real computation of the Ramo theorem.

Single carrier motion and current

Since the current of one e- is higher than the one of one h due to the difference in mobility, normally we try to design a device that uses mainly e-. E.g. if the light is at short wavelength, we design a p over n junction to be sure that all the light is absorbed at the beginning and the electron is the carrier we are interested in. Instead, if the light is absorbed at the end, because e.g. it is long wavelength, then the holes are important, and we can design n over p devices.

Saturated speed of the e- is in the order of 10 ps/um.

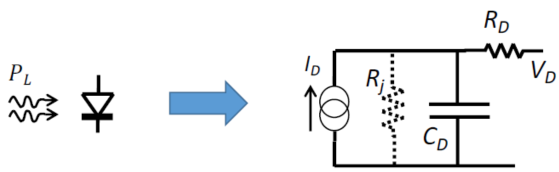
- The duration of a single-carrier pulse is given by the **transit time T_t** of the carrier in the depleted region. At saturated velocity it is quite short: in Silicon the carrier travel takes ≈**10ps/μm**, that is, with w_d = 1 ÷ 100μm it is T_t = 10ps ÷ 1ns.
- The single-carrier pulse duration thus depends on the position of carrier generation. Rigorously, the waveform of the current due to a fast multi-photon pulse is not the convolution of the optical pulse with a standard carrier response: it is a **more complex computation** that depends on the spatial distribution of absorbed photons.
- However, convolution with a suitable standard single-carrier response gives the waveform with approximation adequate for most cases, at least for times longer than the carrier transit time.
- **A simplifying and conservative approximation currently employed for Silicon PDs assumes as standard the response to an electron that crosses all the depletion layer.**

Finite width of response implies <u>low-pass filtering in light-to-current transduction</u>: it's a mobile-mean over time $T_t = w_d/v_{sn}$, with upper band-limit $1/2T_t = v_{sn}/2w_d$.

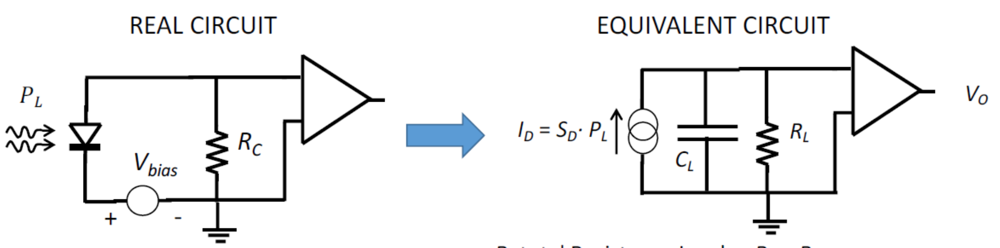
Note the w_d trade-off: long w_d is required for high quantum efficiency at long wavelength λ, short w_d for ultrafast time response. Remark, however, that this is valid for front-illuminated junction and not with side illuminated junction

We can make an assumption: we consider the pn junction as a LP filter for the light (mobile mean filter) with a band limit 1/Tt where Tt is the transit time of the carrier in the whole depleted region.
With this definition we are 'complicating' the tradeoff. The depleted region is a tradeoff between power consumption and detection efficiency, now if we increase it, we are increasing Tt and so reducing the BW.
As for the noise, it is thermal generation density times area times depleted region, so increasing it we are increasing also the noise (besides the power consumption). The depleted region is a number we have to choose depending on the application.

PD EQUIVALENT CIRCUIT



- $I_D = S_D \cdot P_L$ photo-controlled generator (S_D radiant sensitivity or responsivity)
- C_D diode capacitance (p-n junction)
- R_D diode **series** resistance (of the input layer and substrate)
- R_j parallel resistance of the reverse biased junction is considered → ∞



R_C circuit input resistance

R_L total Resistance Load ≈ $R_C + R_D$
C_L total Capacitive Load = $C_D + C_S$ (stray) + C_A (amplifier)

the load circuit is a low-pass filter with time constant $R_L C_L$ in the transfer from detector current I_D to output voltage V_O

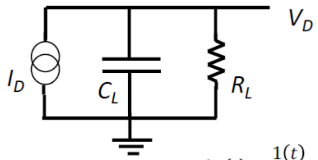
Nn m of the PD. At this point, the input resistance can be maximized, and so also the SNR. However, probably it is not always good to have a high resistance. We are interested in high impedance frontend if we want to maximize the SNR, and this is always true.

However, this is not the path to take if we want to maximize the dynamic response.

PD DYNAMIC RESPONSE

In summary, the PD dynamic response is limited:

1. By the light-to-current transduction, with pulse response $h_D(t)$ of finite-width T_t, well approximated by a rectangular pulse.
2. By the load circuit, with δ-response $h_L(t)$ of finite-width $T_L \approx R_L C_L$



$$h_L(t) = \frac{1(t)}{C_L} \exp\left(-\frac{t}{R_L C_L}\right)$$

The δ-response $h_P(t)$ in the transfer from light power to detector voltage results from the convolution of the two

$$h_P(t) = h_D(t) * h_L(t)$$

Hence the width T_p is the quadratic addition of the two

$$T_P = \sqrt{{T_t}^2 + {T_L}^2} = \sqrt{{T_t}^2 + {R_L}^2 {C_L}^2}$$

For exploiting well the fast response $h_D(t)$ of the PD current, the load circuit does not need to have much faster response, but just comparable or slightly better

$$T_L = R_L C_L \le T_t$$

As soon as we choose the detector, we fixed the transit time. So how can I minimize the RC of the circuit to maximize the dynamic response. We want a small R to maximize the dynamic performances, but if we do so we worsen the SNR → tradeoff between sensitivity and speed.

However, we want to maximize the speed. We want RC lower than the transit time, and the transit time is fixed. The unknown are area and the resistance, then the saturated speed is a constant .

For a PD in planar Silicon with depletion layer w_d and circular area A of diameter D

$$C_D = \varepsilon_{Si} \frac{A}{w_d} \qquad T_t = \frac{w_d}{v_{sn}} \approx w_d \cdot 10 \frac{ps}{\mu m}$$

Assuming (quite optimistically) that the load capacitance be given only the junction $C_L \approx C_D$ and applying the condition $R_L C_L \le T_t$ we get

$$A \le \frac{{w_d}^2}{v_{sn}} \frac{1}{R_L \varepsilon_{Si}} \qquad \text{that is} \qquad D \le w_d \sqrt{\frac{1}{\pi v_{sn} R_L \varepsilon_{Si}}}$$

In wide-band operation the load resistance R_L is small, but is not much less than 100 Ω (diode resistance ≈ some ten Ohm and characterisic resistance of wide-band circuits 50÷75Ω). For exploiting well the fast response limited by the transit time, with R_L = 100 Ω, $\varepsilon_{Si} \approx$1,06pf/cm, $v_{ns} \approx 10^7$ cm/s , the limit to the size of sensitive area is

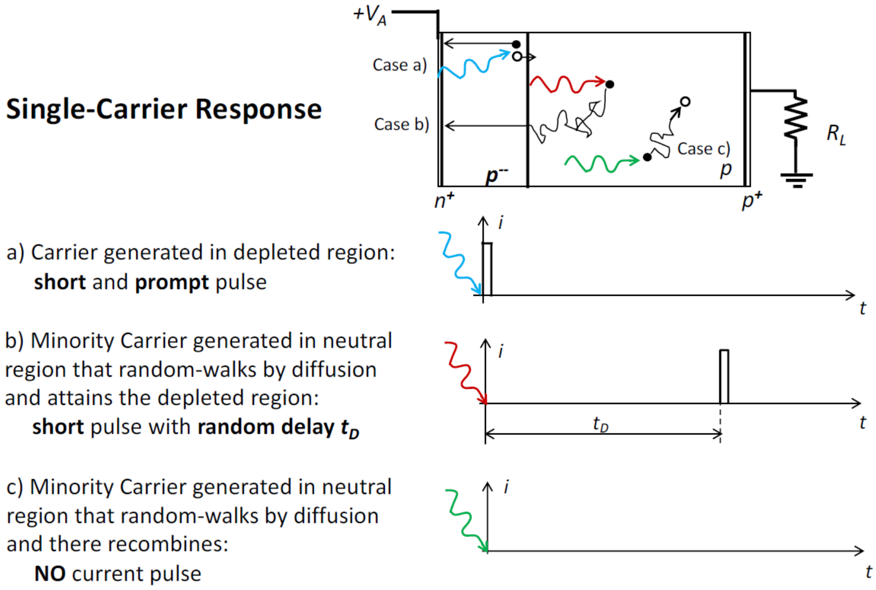
$$D \le 12{,}5 \cdot w_d$$

In the design of detector devices, the selected depletion layer depth w_d depends on the wavelength of interest and on the photon detection efficiency sought; it actually ranges from 1µm to about 100µm.
The area of fast semiconductor photodiodes thus is small in all cases: as w_d ranges from 1µm to 50µm the limit diameter correspondingly ranges from 25µm to 1,25mm

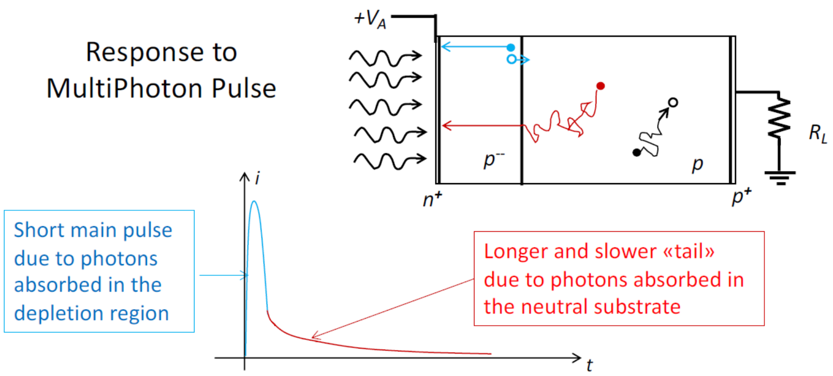
In the end we get that, to have a time response limited only by the transit time and not the RC of the circuit, the diameters must be 12,5 times lower than the depleted region, and this is not good (60 um of diameter). If so, it is really difficult to focus the light on the detector.

CARRIER DIFFUSION EFFECTS

If a single photon is absorbed in the depleted region, it gives us a current. instead, if we are in the neutral region, in theory we don't have any contribution to the current, but in reality a carrier generated in the neutral region doesn't give us any contribution to the current if it remains in the neutral region. In fact, it can diffuse and enter in the depleted region. At that point it acts as a carrier in the depleted region. This eventually happens after a delay that is random.

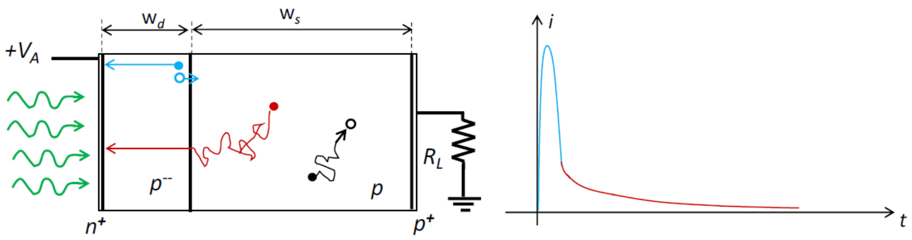


So the response has a gaussian shape with a tail that gives the sum of all the photons absorbed in the neutral region that with a random delay reach the depleted region.
If I'm interested in the detection efficiency, the tail is good because the detection efficiency is higher.



The shape and relative size of the «diffusion tail» are established by the photogeneration and by the diffusion dynamic of minority carriers in neutral regions. They strongly depend on the PD device geometry, on the material properties in the neutral regions (diffusion coefficient and minority carrier lifetime) and on the space distribution of the absorbed photons, hence on the photon wavelength.

Instead, the tail is a problem if we have a high rate of pulses. With one single photon I'm generating one single carrier, so the single electron response gives us the probability of having a certain current depending on where the photon is absorbed.

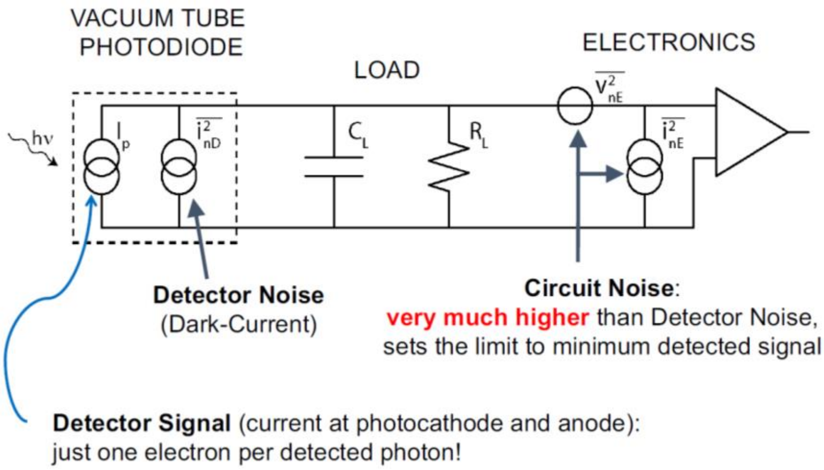


The «diffusion tail»:

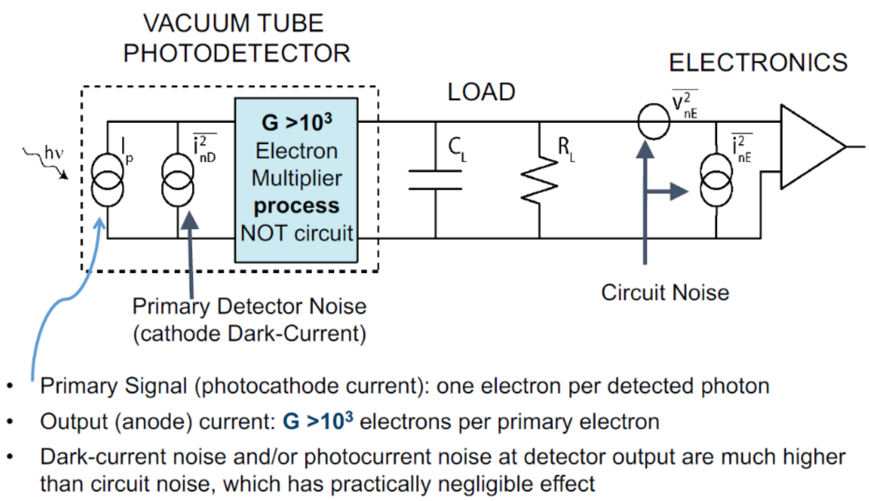
- increases the photon detection efficiency, by bringing to the output a contribution from photons absorbed in a neutral region
- downgrades the detector dynamic response, since the diffusion tail is definitely longer than the prompt pulse
- The time span of the tail increases with the thickness w_s of the neutral substrate and with the minority carrier lifetime, which is longer at lower doping level.
- In Si-PD the tail can be quite significant, ranging from a few 100ns with thick layer (w_s >100µm) and low doping (≈10^{14}/cm^3) to a few 100ps with thin layer (w_s≈1÷2µm) and moderately high doping (≈10^{16}/cm^3).

PHOTOMULTIPLIER TUBE – PMT

In almost all the cases the noise of the PMT and sensor is negligible with respect to the noise of the electronics. If so, is there a way to improve the situation starting from the fact that the noise of the PMT is very low? Yes, we can add a gain.

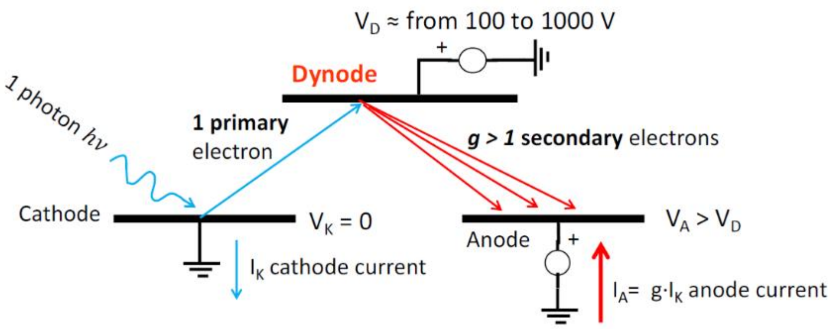


We are applying the gain between the PMT and the amplifier. We are actually amplifying both the signal and noise, so we are not changing the SNR of the PMT itself, which is however very high.
If we applying a gain we can improve the signal and at the sme time we are also increasing the noise, but since the noise is small, even if amplified it could be negligible than the one of the preamplifier. Our goal is to reach a noise that, amplified, is higher than the one of the preamplifier.



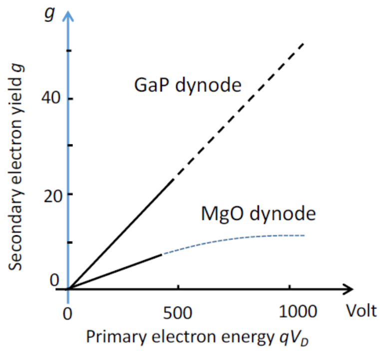
The gain is in the order of 10^3 – 10^6. Of course I cannot amplify with an amplifier, or the situation is not changing. So we need to create a gain without adding a noise, which happens with an amplifier.
The idea is to modify the structure of the PMT.

So far we have a cathode that generates a free e- due to the impinging photon. With an electric field it reaches the anode and generate current. now we add a dynode, which, if hit with an e- with high energy, generates more than 1 electron. Of course, we need to accelerate the first electron to reach high energy, so we need high voltages, from 100 to 1000V.



- A primary electron is emitted in vacuum with very little kinetic energy $E_e < 1eV$
- Driven in vacuum by a high potential difference (some 100V), it impacts with high energy on a **dynode** (electrode coated with suitable material, see later)
- Energy is transferred to electrons in the dynode; some of them gain sufficient energy to be emitted in vacuum; ***g > 1*** is the **yield** of secondary electrons per primary electron

DYNODE MATERIALS



Secondary emitter coatings with ordinary yield:

- MgO Magnesium Oxide
- Cs_3Sb Cesium Antimonide
- BeO Beryllium Oxide
- Cu-Be Copper-Beryllium alloys

Secondary emitter with high yield (due to NEA negative electron affinity, see slide 26 in PD2) :

- GaP Gallium Phosphide

- In the normal working range up to ≈500V, the emission yield *g* is **proportional** to the accelerating voltage V (i.e. the primary electron energy) $g = k_s V_D$
- At higher voltage *g* rises slower and tends to saturate (energy is transferred also to electrons in deeper layers, which have lower probability of escape in vacuum)
- In the linear range ordinary emitters work with ***g* values from ≈1,5 to ≈7** and GaP dynodes *g* values from ≈5 to ≈25
- **GaP dynodes are more costly and delicate**, require special care in operation and their yield tends to decrease progressively over long operation times

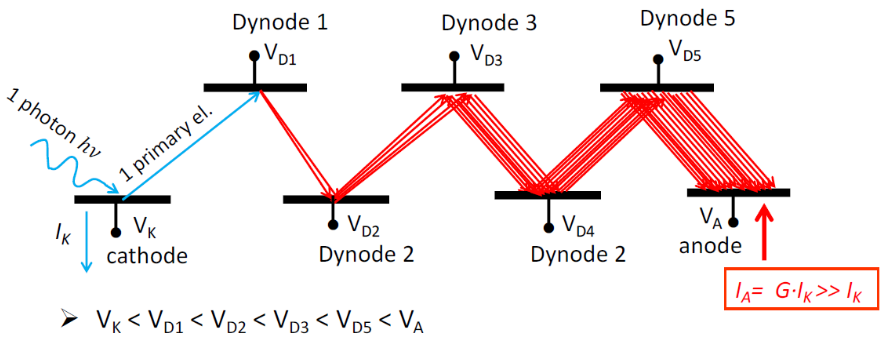
With standard material it is linear (the secondary yeld) and then tends to saturate.
To significantly increase the gain, the idea is to use a chain of dynodes.

Drawbacks

For each dynode we need some hundreds volts of bias one paired with the other, so at the end of the chain we have a very high voltage. For each dynode the gain is proportional to the bias voltage, and the overall voltage is the product of all the intermediate gain.
Furthermore, the structure in the image below is a mechanical structure, the dynode must be placed to focus the electrons from one dynode to the other, so we have to mechanically align them to have a very good gain.

So the structure of the PMT becomes complicate because of this chain of dynodes.

Sketch of the Principle (example with 5 dynodes)

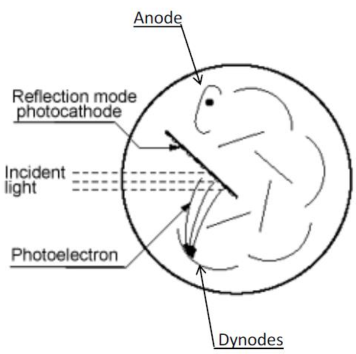


- $V_K < V_{D1} < V_{D2} < V_{D3} < V_{D5} < V_A$
- Electron optics (i.e. potential distribution) **carefully designed to lead the electrons** emitted from each electrode to the next one
- $g_r > 1$ secondary electron yield of dynode r
- $G = g_1 \cdot g_2 \cdot g_3 \cdot g_4 \cdot g_5$ **overall multiplier gain**

  that is, $G=g^5$ with equal stages $g_1=g_2=....=g$

PMT STRUCTURES

PMTs with side-window and opaque photocathode

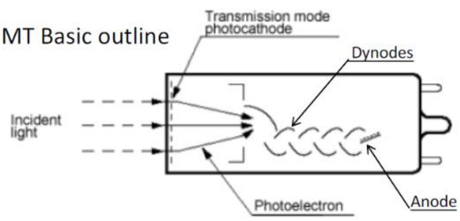


The basic structure of Photomultiplier Tubes with discrete dynodes and electrostatic-focusing was first demonstrated in 1937 by the RCA Laboratories; in the following decades it was progressively improved and developed by various industrial laboratories (RCA, DuMont, EMI, Philips, Hamamatsu...)

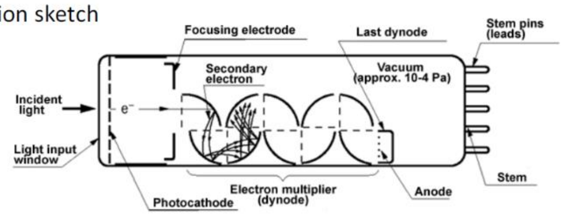
Moreover, a problem is that <u>if we have a magnetic field, it could change the direction of e- and so the gain of the system</u>, and this is not good. So EMI are a problem for PMT.

PMTs with end-window and semitransparent photocathode

PMT Basic outline



PMT Operation sketch



The PMT is able to detect a single photon (very high sensitivity), and the noise after the amplification is very low. The problem is that it is bulky, fragile (made of glass), sensitive to the magnetic field.

GAIN

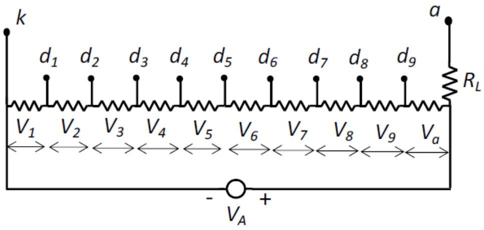
- PMTs can have high number *n* of dynodes (from 8 to 12) and attain high gain G.

  With n equal dynodes it is $G = g^n$; e.g. with 12 dynodes $G=g^{12}$

  $G = 10^4$ with g = 2,2

  $G = 10^5$ with g = 2,6

  $G = 10^6$ with g = 3,2
- G is controlled by the dynode bias voltage, which regulates the dynode yield g
- A single supply is usually employed, with high voltage V_A typically from 1500 to 3000 V. The dynode voltages are obtained with a voltage-divider resistor chain; the potential difference V_j between two dynodes *j* and *(j-1)* is a preset fraction f_j of the supply V_A



$$\boxed{V_j = f_j\, V_A}$$

In order to obtain a very large gain we have to increase the bias voltage between each dynode and the following one. To do so, we create not 12 different bias voltages, but we use a voltage divider.
The problem is that the power dissipation is very huge, which is not our goal.

The problem is that if we are extracting electrons from the dynodes to generate electrons, the biasing of the dynodes themselves is changing because we have a current flowing in the biasing resistor. Since the bias voltage is connected to the gain in a linear way, we are also changing the gain and the signal.
The good thing is that this happens only if we have a pulse of light.

PMT GAIN REGULATION AND STABILIZATION

- The supply voltage V_A thus rules the yield g_j of every dynode $g_j = k_S V_j = k_S f_j V_A$

  and the total gain $G = g_1 g_2 \dots g_n = k_S V_1 \cdot k_S V_2 \dots k_S V_n = k_S^n f_1 f_2 \dots f_n \cdot V_A^n$

  which increases with V_A **much more** than linearly

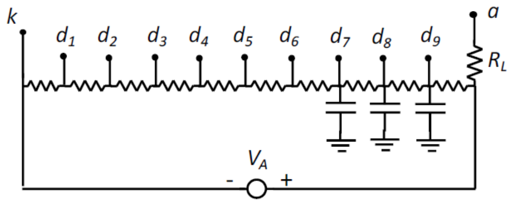
  $$\boxed{G = k_S^n f_1 f_2 \dots f_n \cdot V_A^n = K_G \cdot V_A^n}$$

  (NB: $K_G = k_S^n\, f_1 f_2 \dots f_n$ is constant, set by the voltage distribution and dynode characteristics)
- The gain G is very sensitive to even small variations of the supply V_A : the relative variations of supply voltage are n-fold amplified in the relative variations of gain

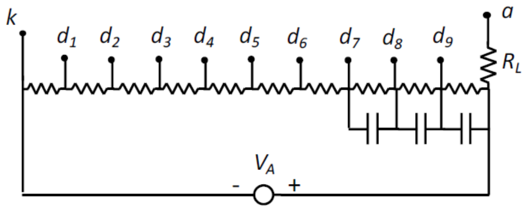
  $$\boxed{\frac{dG}{G} = n \frac{dV_A}{V_A}}$$
- Consequently, tight requirements must be set to the stability of the high voltage V_A versus ambient temperature and/or power-line voltage variations.
  e.g. getting ***G* stability better than 1%** for a PMT **with n=12 dynodes** requires a high voltage supply ***V_A* better stable than 0,08 %**

To solve the issue of the gain we can add capacitances. When a pulse of light comes, we have the generation of an electron but this electron flows through the capacitance and then the system goes back to the steady state value.

- The parameter values in the PMT operation must be carefully selected for exploiting correctly the PMT performance. We will point out some main aspects and call the **user attention on warnings reported in the manufacturer data sheets**.
- For **limiting self-heating of voltage divider** below a few Watt, the divider current must be < 1 mA, hence total divider resistance must be at least a few MΩ.
- In order to avoid nonlinearity in the current amplification, variations of dynode voltages caused by the PMT current should be negligible. The PMT output current must thus be less than 1% of the divider current, i.e. typically a few μA.
- This limit is acceptable for DC current, but not for pulsed optical signals. However, fast transients of dynode voltages can be limited by introducing in the last stages capacitors in low-pass filtering configurations, as sketched in the examples



The problem with having the capacitances to ground is that we want to use the capacitance on a fast electrical pulse, and we cannot make fast capacitances with 3kV of bias voltage, because on one side the capacitance has ground, on the other side a huge voltage, so it cannot be a fast responsive capacitance. So their arrangement must be changed.



- **Space-charge effects may cause nonlinearities** in the amplification of fast pulsed signals. A high charge of the signal itself can significantly reduce the electric field that drives the electrons: the higher is the pulse, the slower gets the electron collection. The pulse shape is more or less distorted, depending on its size
- Nonlinearity can occur also if the voltage signal developed on the load is high enough to reduce the driving field from last dynode to anode
- **Magnetic fields have very detrimental effect**: the electrons traveling in vacuum are deviated and the operation is inhibited or badly degraded. With moderate field intensity, magnetic screens (Mu-metal shields wrapped around the vacuum tube) can limit the effects; with high intensity fields PMT operation is actually impossible
- **PMTs are fairly delicate** and subject to fatigue effects and their operation is prejudiced by mechanical vibrations

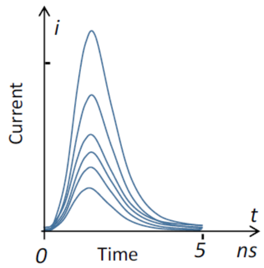
Now each capacitance has the bias voltage just between two dynodes, so it can be fast. However, the issue related to the magnetic field is still there.

SINGLE ELECTRON RESPONSE – SER

The gain is not constant; from the cathode we have one electron, which goes to the dynode, and the dynode generates a number of e- that every time changes, it is a statistical number of electrons generated. So the gain of each dynode changes a little. Putting all the variations together in a chain, the SER can change a lot in terms of intensity.

However, the detector is still very fast (ns), much faster than a photocathode.

- The PMT output is superposition of elementary current pulses that correspond to single electrons emitted by the cathode, called **Single Electron Response (SER)** pulses.
- SER current pulses are fast (a few nanosecond width) and fairly high (pulse-charge Gq from 10^5 to 10^6 electrons). **They are remarkably higher than the noise** of fast circuits; with PMT weakly illuminated they are well observable on the oscilloscope screen and each of them corresponds to the detection of a single photon.
- The SER current pulses observed have all equal pulse shape, but **randomly varying pulse-amplitude**; i.e. G is not constant, but statistical

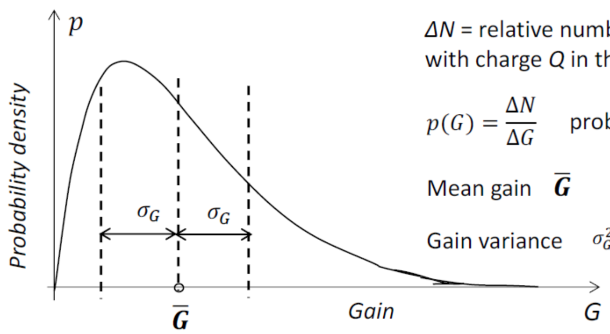


- The random fluctuations of G are due to the statistical nature of secondary electron emission
- Since the SER charge is much higher than the minimum measurable detector pulse*, the statistical distribution *p(G)* of the gain *G* (probability density of *G* value) can be directly collected by measuring and classifying the pulse-charge of many SER pulses.

Applying the Ramo theorem, we have to ground all the dynodes, and the reference electric field is only between the last dynode and the anode. However, **if the gain is not constant, the signal also is not constant**. This gain applies both on the signal and noise.

Fluctuation of the signal is like having a new source of noise that we want to avoid. We can manage this noise like if we have a system that amplifies, and the noise at the output is the noise at the input times the gain squared. If the gain is changing, we have to add a factor, the excess noise factor F, which takes into consideration the fact that the noise is not constant.

Statistical distribution of the PMT gain



ΔN = relative number of SER pulses observed with charge *Q* in the interval *qG ± qΔG/2*

$p(G) = \frac{\Delta N}{\Delta G}$ probability density of *G*

Mean gain $\overline{G}$

Gain variance $\sigma_G^2 = \overline{(G-\overline{G})^2} = \overline{G^2} - (\overline{G})^2$

- The plot above sketches the typical appearance of the statistical distribution *p(G)* of the PMT gain *G.*
- For different PMT models and different operating conditions (bias voltage distribution on dynodes; temperature of operation; etc.) remarkably different *p(G)* are observed. The distributions are roughly akin to gaussian, but skewed toward high *G* values.
- The main parameters to be considered for analyzing the PMT operation are mean gain $\bar{G}$, gain variance σ_G^2 and relative variance $v_G^2 = \frac{\sigma_G^2}{(\bar{G})^2}$

- Emission of primary electrons from cathode is a process with Poisson statistics, i.e. mean number N_p , variance $\sigma_p^2 = N_p$ and relative variance $v_p^2 = \frac{\sigma_p^2}{N_p^2} = \frac{1}{N_p}$
- Emission is followed in cascade by statistical multiplication with fluctuating G
- The mean of the cascade output is $N_u = N_p \cdot \bar{G}$ (two independent processes)
- The Laplace theory of probability generating functions shows that the relative variance $v_u{}^2$ of the output of a cascade is sum of the relative variance of every stage in the cascade divided by the mean value of all the previous stages. In our case:

$$v_u^2 = \frac{\sigma_u^2}{N_u^2} = v_p^2 + \frac{v_G^2}{N_p} = \frac{1}{N_p} + \frac{v_G^2}{N_p} = \frac{1}{N_p}(1 + v_G^2)$$

- The variance σ_u^2 thus is

$$\sigma_u^2 = N_p^2 \overline{G}^2 v_u^2 = N_p \overline{G}^2 (1 + v_G^2) = \sigma_p^2 \, \overline{G}^2 (1 + v_G^2)$$

In conclusion, the PMT :
1) amplifies the input variance by the square gain $\bar{G}^2$, like an amplifier and
2) further enhances it by the **Excess Noise Factor F** due to the gain fluctuations

$$\boxed{\sigma_u^2 = \sigma_p^2 \cdot \overline{G}^2 \cdot F} \quad \text{with} \quad \boxed{F = 1 + v_G^2 > 1}$$

The result is that, if we have some dynodes, the final variance of the gain is the variance of each single dynode times the excessive noise factor. Normally, F = 1 in a PMT.

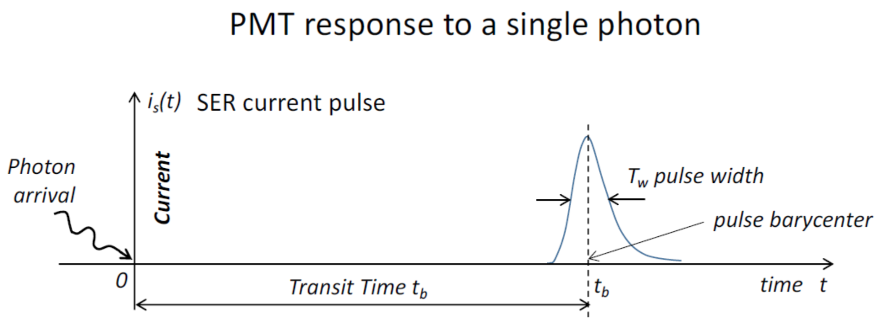
- A PMT amplifies by $\bar{G}^2$ the input noise like an amplifier and further increases it by the **Excess Noise Factor *F* :** $\sigma_u^2 = \sigma_p^2 \cdot \bar{G}^2 \cdot F$
- We will see that it is ***F* ≤ *2* for most PMT types and *F* is close to unity for high quality PMT types**. The factor of increase of rms noise is always **moderate** $\sqrt{F} \leq 1{,}4$ and often near to unity. Reasonably approximated evaluations can be obtained by neglecting the excess noise, i.e. with *F=1*.
- As modern alternative to a PMT, one could propose a vacuum tube photodiode coupled to a high-gain and low-noise amplifier chip, possibly with amplifier chip inside the vacuum tube. It would offer practical advantages: more simple, rugged and compact structure, lower operating voltage, etc..
- In fact, a PMT outperforms such «photodiode-with-amplifier-inside» by detecting optical signals smaller by orders of magnitude. We can better understand the matter by gaining a better insight about how these devices work.

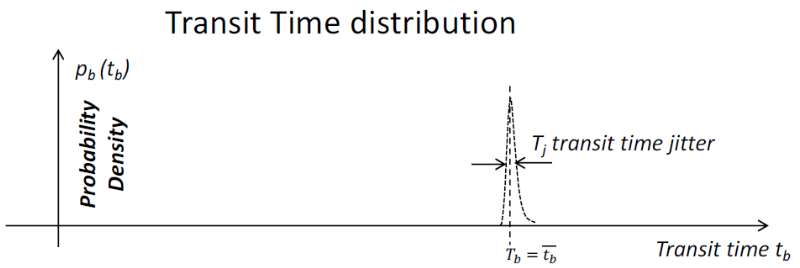
One idea could be to include an amplifier directly inside the device to obtain a better amplification without the F problem. The thing is that the good thing about PMT is how the amplification is made with dynodes.
Without any signal, the amplifier has a certain noise at the output, which is not 0. The noise of a PMT at the output if we have 0 signal we have no noise due to the amplification (of course we would have the noise of the cathode, but we are dealing with amplification noise).

So in this way we can get a very high gain without adding significant noise.

- In the amplifier a **signal gains energy from the power supply by modulating the bias current in transistors**, which must be active all the time. The amplifier **noise sources are always active** (shot noise of transistor bias current; Johnson noise of resistors)
- In a PD-amplifier combination it is the amplifier noise that sets the limit to the minimum measurable signal, since it is much higher than the photocathode dark-current noise
- In a PMT, the **electrons of the signal gain energy directly from the voltage supply**: the bias voltage accelerates them and the kinetic energy gained is exploited in the impact to generate other free electrons. There is **no bias current** in the multiplier chain, the current flows only when electrons are injected from the cathode.
- In a PMT there are **no noise sources in the dynode chain**; the minimum signal is limited by the dark-current noise and/or the photon-current noise at the cathode.
- The cathode noise is indeed slightly increased by the gain fluctuations in the dynode chain, but in practice this is always a minor effect and often it is negligible.

DYNAMIC RESPONSE OF PMTs





The one in the image is the single electron response of a PMT. We notice that there is a delay between the peak of the single electron response and the photon arrival. This is not strange, because the anode can see the electrons only when they reach the last dynode, and this is the reason for the delay.

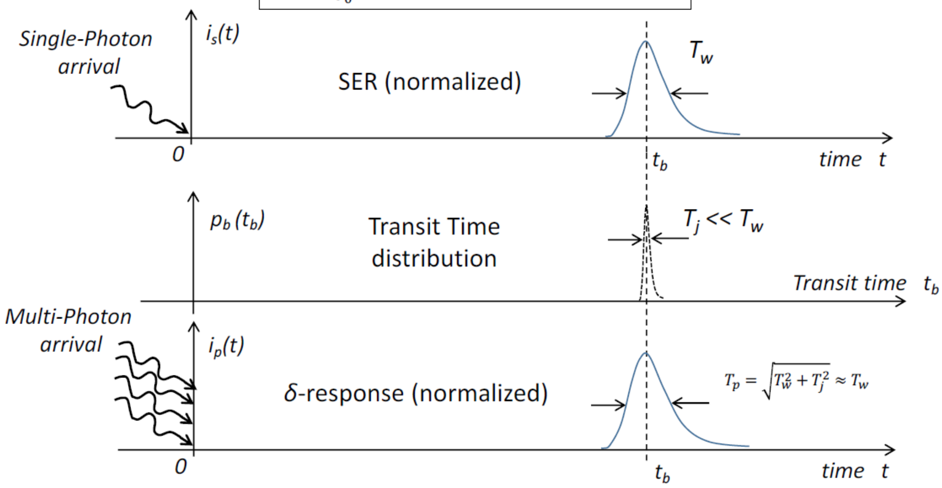
We are not concerned about the delay because it is an offset in time of only few ns, and already connecting with a cable the device to the PS generates ns time of delay.
The problem is that the offset in time is not always the same, it is a jitter that can depend on the trajectory of the first electron to reach the dynode.

- Differently from vacuum tube photodiodes, in PMT the rise of a SER current pulse is delayed (from ≈10ns to some 10ns dependent on PMT type and bias voltage) with respect to the photon arrival. The dynodes electrostatically screen the anode, so that only electrons traveling from last dynode to anode induce current (Shockley-Ramo theorem).
  The **PMT transit time t_b** is defined as the delay of the **pulse barycenter**.
- **The transit time t_b randomly fluctuates from pulse to pulse, with a transit time jitter T_j** (full-width at half maximum FWHM of the t_b distribution) from a few 100ps to a few ns depending on PMT type and bias voltage. T_j is due to the statistical dispersion of the electron trajectories in the *first stages of the multiplier*.
- **The SER pulse width T_w** (FWHM from a few ns to various ns, depending on PMT type and bias voltage) is always wider than the transit time jitter: $T_w \approx$ 5 to 10 times T_j. It is due to the statistical dispersion of the electron trajectories *in all the multiplier*.
- T_w has very small fluctuations, practically negligible

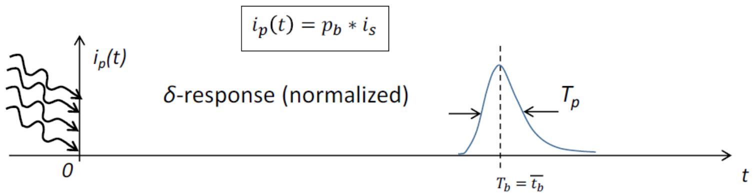
In the real world, we have in the single electron response there is the convolution of two effects. Firstly

PMT response $i_p(t)$ to a multi-photon δ-like light pulse:
derived from 1) SER pulse waveform and 2) transit time distribution

$$i_p(t) = \int_0^{\infty} p_b(t_b)\ i_s(t-t_b)dt_b = p_b * i_s$$



the real single electron response, when we consider that is no jitter on the transit time, then this electron response will move on the left or right depending on the trajectory of the electron for each single pulse. If we convolve these two effect we get the real electron response. The good thig is that the jitter of the transit time is much smaller than the ideal single electron response, so we don't notice any problem on the jitter.



- The δ-response is a convolution $i_p = p_b * i_s$, hence its FWHM T_p is quadratic sum of FWHMs T_w and T_j of the components

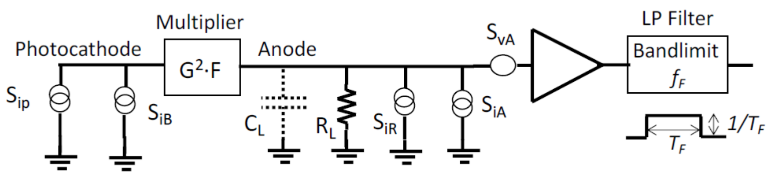
$$T_p = \sqrt{T_w^2 + T_j^2}$$

- since T_j / T_w is small (from 0,1 to 0,2) the width of the δ-response is practically equal to the SER current pulse width $T_p \approx T_w \left[1 + \frac{1}{2}\left(\frac{T_j}{T_w}\right)^2\right] \approx T_w$
- The finite SER pulse width establishes a finite bandwidth f_p for the PMT employed as analog current amplifier

$$f_p = \frac{1}{k_a T_w}$$

(the coefficient k_a is from ≈3 to ≈10, depending on the SER pulse waveform)

SNR AND MINIMUM MEASURABLE SIGNAL



- n_p photoelectron rate → $I_p = n_p q$ photocurrent
- n_D dark electron rate → $I_D = n_D q$ cathode dark current
- n_b electron rate due to photon background → $I_b = n_b q$ photon background current
- $n_B = n_D + n_b$ total background electron rate → $I_B = n_B q$ total background current

Noise sources :

- at cathode: $S_{ip} = 2qI_p = 2q^2 n_p$ photocurrent noise, **increases with the signal**
- at cathode: $S_{iB} = 2qI_B = 2q^2 n_B$ background noise, **independent from the signal**
- at anode: resistor load noise S_{iR} and preamplifier noise S_{iA} and S_{iR}

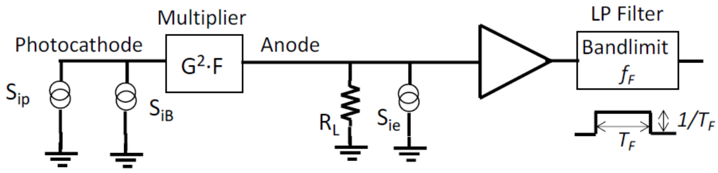
Let's deal with S/N and minimum measurable signal in the basic case:
constant signal current I_p and **low-pass filtering** (typically by Gated Integration)

We have the current generator of the cathode on the left, then the current noise of the cathode, then the gain with the excessive noise factor, the load, the noise spectral density of the resistance and noises of the amplifier. Then we have a LP filter to limit the BW of the system, if the amplifier is ideal.

We will consider the photon electron current, which is the number of electrons times the charge of the electrons. We count the number of photons because the current is so low we can do this. Since we have to compare signal and noise, we will count the number of electrons also for noise.

With a photosensor we have to add one more noise; we have the current noise, voltage noise, noise of the cathode (shot noise); but in this case the signal has a fluctuation, a Poisson one, so we have signal associated noise. It is the first time that the noise is directly related to the signal.

Rl and Cl give us a tau, but if the pole is much higher than the LP filter at the end, we can neglect the filtering action of Rl and Cl. Of course we neglect only the filtering action, not the noise..

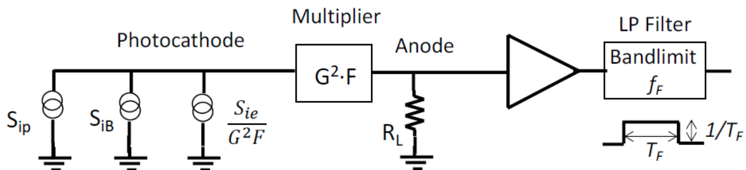


We consider cases with wide-band load, i.e. with $1/4R_LC_L >> f_F$, such that

a) the filtering effect of C_L is negligible
b) the circuit noise can modeled simply by a current generator

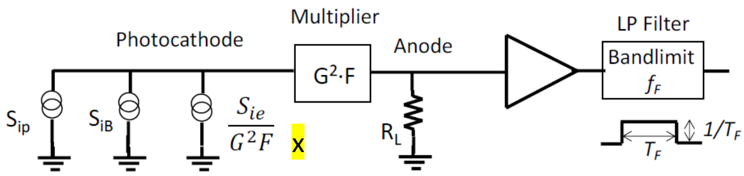
$$S_{ie} = S_{iA} + S_{iR} + \frac{S_{vA}}{R_L^2}$$

which can be referred back to the input (at the photocathode) as S_{ie}/G^2F



We want to place all the sources of noise in only one single current generator because we want to input-refer the output noise. If so, we can study the problem at the cathode level.

We can model noise x as shot noise, even if it is not shot noise, this because all the other sources of noise are shot. To do so, we can create a current that is the spectral density divided by two times the charge. It is a 'fake' current. at this point we have only shot noise on the left, and we can compare signal and noise, directly comparing the current of the signal and of the noise.

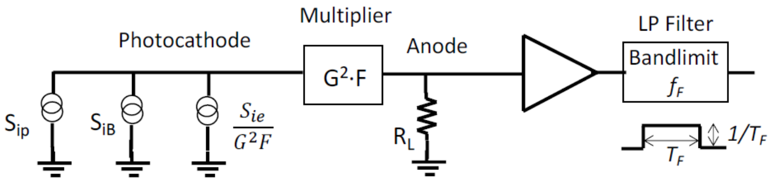


- The circuit noise S_{ie} can be modeled by a shot current **at the anode:**
  $I_e = S_{ie}/2q$ with electron rate $n_e = I_e/q = S_{ie}/2q^2$
- With wide band preamplifier and low resistance $R_L \approx$ few kΩ the circuit noise typically is $\sqrt{S_{ie}} \approx 2\, pA/\sqrt{Hz}$ or more. The equivalent shot electron rate is $\boldsymbol{n_e \approx 10^{14}}$ ***el/s*** **or more**
- Referred to input (cathode), the circuit noise is modeled by a shot current with **reduced** electron rate n_e/FG^2. For instance, with G= 10^6 it is $\boldsymbol{n_e/FG^2 \approx 100}$ ***el/s***
- The circuit noise referred to the input added to the background noise $S_{iB}=2qI_B=2q^2n_B$ gives the **constant** noise component (i.e. **NOT** dependent on the signal)

$$S_{iB} + \frac{S_{ie}}{G^2F} = 2qI_B + \frac{2qI_e}{G^2F} = 2q^2\left(n_B + \frac{n_e}{G^2F}\right)$$

In a lot of cases, the noise associated to the anode is negligible with respect to the noise associated to the dark current, to the signal and to the background noise, which are the sources of noise we have at the cathode level. All these can be modelled as shot noise, and we can lump them in one single current generator.

So I have the signal, shot noise associated to the signal and shot noise independent from the signal. Since the gain of the PMT is high, I can neglect the independent shot noise, but this is true up to a certain point; in fact, also PMT has noise (dark current and background) that is independent on the noise, so we would still have the same problem.



$$S_{iB} + \frac{S_{ie}}{G^2F} = 2qI_B + \frac{2qI_e}{G^2F} = 2q^2\left(n_B + \frac{n_e}{G^2F}\right)$$

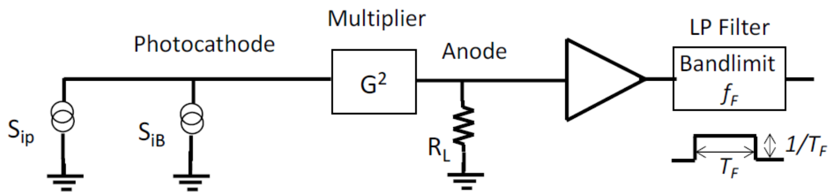
- The role of the circuit noise is assessed by comparing it to the constant noise source of the PMT, the background noise $S_{iB}=2qI_B=2q^2n_B$
- The background electron rate at the cathode n_B may vary ***from a few el/s to a few 10^6 el/s***, depending on the photocathode type and operating temperature and on the background light level (see Slides PD2)
- In **most cases** of PMT application it is $n_B >> n_e/G^2F$: the equivalent electron rate n_e/G^2F is **totally negligible** with respect to n_B , the circuit noise plays no role
- In cases with moderate gain G and/or very low dark current the circuit noise contribution may be significant and is very simply taken into account, by employing the resulting density of constant noise component in the evaluation

MEASURABLE MINIMUM SIGNAL

For the sake of simplicity in the following computations we consider:

a) **negligible circuit noise.** Anyway, we know when it must be taken into account and how to do it, by considering an increased constant component of noise.

b) negligible excess noise, i.e. ***F* = 1**. Anyway, cases with non-negligible F > 1 can be taken into account simply by introducing the factor $\sqrt{F}$ to decrease the S/N and increase the noise variance and the minimum signal computed with F=1.



$$\boxed{\frac{S}{N} = \frac{I_p}{\sqrt{S_{ip}f_F + S_{iB}f_F}} = \frac{I_p}{\sqrt{2qI_pf_F + 2qI_Bf_F}}}$$

The minimum signal $I_{p,min}$ is reached when S/N = 1 : we will see that the result markedly depends on the **relative size of constant noise vs photocurrent noise**

Sip is the noise associated to the signal, Sib is the noise independent of the signal, and it is shot noise due to dark current, shot noise of the background and not shot noise associated to the amplifier divided by the gain. But we can always take the total spectral density divided by 2*q to get an equivalent current. We notice that the current Ip at the top of the SNR is also at the bottom of the ratio.

We can solve this equation for a SNR = 1 to retrieve Ip (Ib is given from the data). On the other side, we can make some hypothesis to be verified. If the final current is so high that the shot current associated to this current is much higher than the term with Ib, I can neglect the term with Ib and the equation is easier to be solved. On the other side, if the final background is much higher, Ip at the denominator can be neglected, and I'm in the classical condition with signal at the top and noise at the bottom of the SNR. Then I have to verify if the hypotheses are correct.

The drawback of making the hypotheses is that if we do the calculations and then we get that they are not consistent, we have to redo the computations.

Minimum signal limited by photocurrent noise

We are in the case of negligible background noise, so we completely remove the part that is independent on the signal because we suppose it is not dominant.

- The simplest **extreme case is with negligible background noise:** only photocurrent noise matters. With noise band-limit $f_F = 1/2T_F$ (GI filtering)

$$\frac{S}{N} = \frac{I_p}{\sqrt{2qI_pf_F}} = \frac{I_pT_F}{\sqrt{qI_pT_F}} = \sqrt{\frac{I_pT_F}{q}} = \sqrt{n_pT_F} = \sqrt{N_p}$$

  $N_p = n_p T_F$ is the **number of photoelectrons** in the filtering time T_F .

- In fact, the S/N can be obtained directly from the Poisson statistics of photoelectrons: with mean number N_p , the variance is $\sigma_p^2 = N_p$ and

$$\frac{S}{N} = \frac{N_p}{\sigma_p} = \frac{N_p}{\sqrt{N_p}} = \sqrt{N_p}$$

- Remark that in this case the noise is **NOT constant,** independent from the signal: as the signal goes down, **also the noise goes down!!**

We find that the SNR is proportional to the rate of photons time Tf, that is the number of photons we are reading with our gain.
np (I/q) is a rate, and Tf is a time, so their product is the total number of photoelectrons I get, Np.

But the rate times a time is a total number of photons.
We are dominated by the shot noise of the signal itself, so by the statistic of the signal itself. If I'm acquiring n photons, n is the signal. Since the light is a Poisson process, if I acquire n photons, its variance is n, so the square root of N is the SNR. This is another way to see the same thing.
The strange thing is that the value sqrt(Np) is not constant.

For a SNR = 1, which is the minimum signal we can detect?

We write the same equation of before, and Nmin = 1 if we are dominated by the statistic of the signal, so it is one photon (or a rate 1/Tf).

- By making lower and lower I_p , **when S/N = 1 the minimum signal $I_{p,min-p}$ is reached**

$$\boxed{\left(\frac{S}{N}\right)_{min} = 1 = \sqrt{\frac{I_{p,\min -p} T_F}{q}} = \sqrt{n_{p,\min -p}\ T_F} \sqrt{N_{p,\min -p}}}$$

- The minimum measurable photocurrent signal $I_{p,min-p}$ corresponds to just **one photoelectron in T_F** , the filter weighting time:

$$\boxed{I_{p,\min -p} = \frac{q}{T_F}} \qquad \boxed{n_{p,\min -p} = \frac{1}{T_F}} \qquad \boxed{N_{p,\min -p} = 1}$$

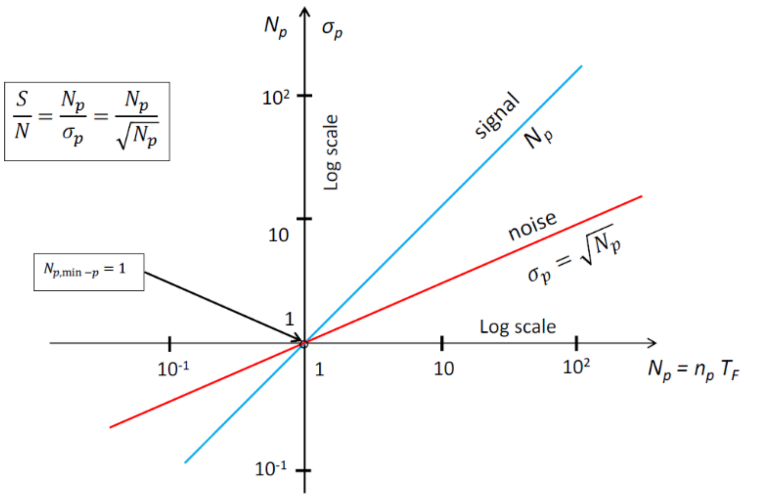
- Observing the complete S/N equation

$$\frac{S}{N} = \frac{I_p}{\sqrt{2qI_pf_F + 2qI_Bf_F}} = \frac{I_pT_F}{\sqrt{qI_pT_F + qI_BT_F}} = \frac{n_pT_F}{\sqrt{n_pT_F + n_BT_F}} = \frac{N_p}{\sqrt{N_p + N_B}}$$

we see that the background noise is truly negligible only if $I_B << I_p$ for any I_p **down to the minimum $I_{p,\,min-p}$** , i.e. only if

$$\boxed{I_B \ll \frac{q}{T_F}} \qquad \boxed{n_B \ll \frac{1}{T_F}} \qquad \boxed{N_B \ll 1}$$

The plot is in log scale, so the SNR is the distance between the two curves, hence a SNR = 1 is where the blue and red curve cross.



Signal measured by **charge**, in terms of **number of photoelectrons $N_p = n_p T_F$**

What if the background noise is not negligible?

Minimum signal limited by background noise

- The **opposite extreme case is with negligible photocurrent noise:** only background noise matters. More precisely, it's the case where the limit current $I_p = I_{p,min-p}$ computed with only the photocurrent noise is much lower than the background current I_B

$$\boxed{I_B \gg \frac{q}{T_F}} \qquad \boxed{n_B \gg \frac{1}{T_F}} \qquad \boxed{N_B \gg 1}$$

- There is now a different **minimum signal $I_{p,min-B}$ limited by the background noise**

$$\boxed{I_{p,\min -B} = \sqrt{\frac{qI_B}{T_F}}} \qquad \boxed{n_{p,\min -B} = \sqrt{\frac{n_B}{T_F}}} \qquad \boxed{N_{p,\min -B} = \sqrt{N_B}}$$

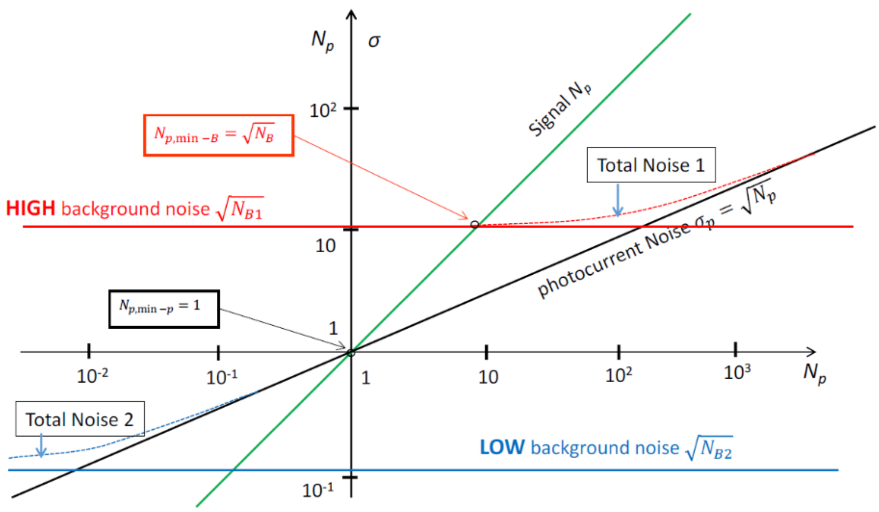
- In **intermediate cases both noise components** contribute to limit the minimum signal, which is computed from

$$\frac{S}{N} = \frac{N_{p,\min}}{\sqrt{NB_{p,\min}}} = 1 \qquad \text{2}^{\text{nd}} \text{ order equation that leads to} \qquad N_{p,\min} = \frac{1}{2}\left(1 + \sqrt{1 + 4N_B}\right)$$

(NB: the other solution is devoid of physical meaning)

If Nb is no more negligible, but much higher than Np, it is the dominant one. Consequently, Np = sqrt(Nb), if we are for SNR = 1.

If neither the background nor the photocurrent noise are dominant we have to solve the complete second order equation. The good thing in making an hypothesis is that we can verify if it is correct or not, while with just the solving of the equation we get a number, and we don't know if it is correct or not.



Signal **charge**, in terms of **number of photoelectrons $N_p = n_p T_F$**

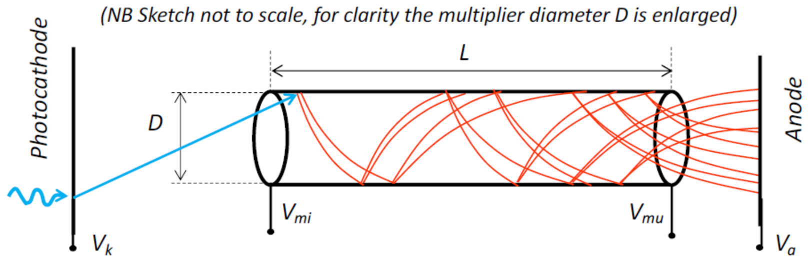
Sqrt(Nb) is a constant because it is not related to the signal. Increasing the background we are shifting the red curve upward, making it dominant over the photocurrent noise.

PMT DEVICE STRUCTURES

Can we improve the device more?

CONTINUOUS CHANNEL MOLTIPLIER CCM

I take a tube and I make a coating of the inside with the material of the dynodes. At this point, as soon as a photon exits the photocathode and enters the tube, we are not losing any generated electron, so maybe the efficiency of the system is better.



In order to get PMTs more simple, compact, robust and less sensitive to mechanical vibrations, minitubular electron multipliers were introduced (in the late years 60's)

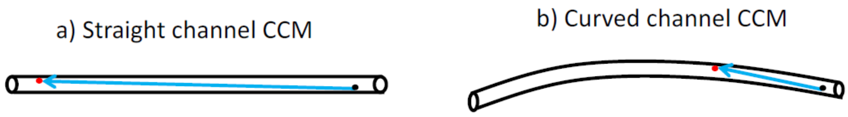
- A special glass **capillary tube with D < 1mm**, called Continuous Channel Multiplier CCM or Channeltron, is at once **voltage divider and electron multiplier**; the inner surface is chemically treated and converted in a semiconductor layer with high resistivity and secondary electron emission yield g ≈ from 1,2 to 3.
- For a given applied voltage the **gain depends on the ratio *L/D***. As *L/D* increases the number of impacts increases, but the yield decreases because the impacting electron energy decreases. Maximum gain is attained with $L/D \approx 50$
- Gain G from 10^5 to 10^6 is attained with applied voltage in the range 2 to 3kV
- **No need to focus electrons** within the multiplier, but the electron optics from cathode to multiplier input must be carefully designed to get good collection efficiency

The problems are multiple. Firstly I have to focus the light inside the tube, the electron has to enter the tube and the diameter and the tube sometimes is not big.
Moreover, if I have an amplification, from one photon we generate an e- that enters the tube and it multiplies. At the end of the tube I have a lot of e- with an extremely high energy, because they are all accelerated. Furthermore, I apply the total bias voltage at the end of the tube, I don't have anymore the voltage divider.

However, high energy e- can create ions due to impact ionization, and the ion has an opposite charge compared with the e- and goes back in the opposite direction, impacting in the tube and generating other e-, so creating a positive feedback, which is increasing a lot the noise.

The idea, instead of taking a straight tube, to use a curved one. Thus a generated ion will impact the tube on a small time in a small portion, so the feedback is limited in time.

Moreover, the tube is very small and we are generating millions of electrons for each photon. Since we have a lot of millions of e- in a small volume, we have space charge that is shielding the electric field, and the only solution we have is to reduce the amount of charge. Of course we cannot do this reducing the gain, because the goal is to have a high gain. So we have to reduce the number of photons → this architecture works if we have few photons.

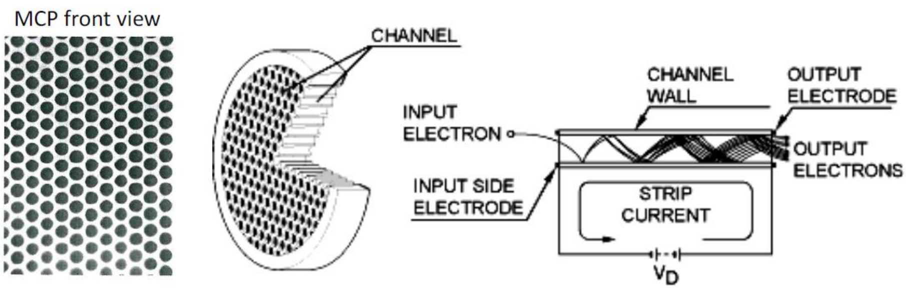


In order to exploit CCMs it is necessary to neutralize the effect of Ion Feedback.

- In the **last part of the channel the density of energetic electrons is high** and creation of free **heavy ions** (ionized atoms) by collision with residual gas molecules (or with the wall material) becomes probable.
- The free ions drift in the field and by impacting on the wall cause a strong emission of electrons. If the impact occurs near the channel input the emitted electrons undergo all the channel multiplication.
- This is a positive feedback effect, which enhances the current amplification in uncontrolled way and may even cause a self-sustaining breakdown current in the multiplier.
- The effect is avoided by bending the axis of the multiplier tube. Due to the large mass and small charge, a free ion has small acceleration in the electric field and its trajectory is almost straight; the ions thus impact in the last part of the channel, hence the emitted electrons undergo undergo a much lower amplification

MICRO-CHANNEL PLATE MLTIPLIER – MCP

Every single channel has the same problem as before of the space charge, but all the photons generate electrons spread over the MCP, so the number of electrons of each tube on average is the total number of electrons divided by the number of tubes, so we can have a small charge effect for each tube but also having a total detector that can manage a very high number of photons.



- For overcoming the CCM limitations, the multiplier concept evolved (in early years 70's) to the MicroChannel Plate MCP, implemented with sophisticated glass technology
- An array of many thousands of multiplier microtubes is embedded in beehive structure into a plate. All channels are biased in parallel with the same high voltage V_D , applied via metal electrodes deposited on the two faces of the plate.
- The MCP has a **planar geometry**, well matched to a planar end-window photocathode; focusing of photoelectrons on the multiplier is simply provided by a high voltage from cathode to multiplier input (**proximity-focusing** geometry)

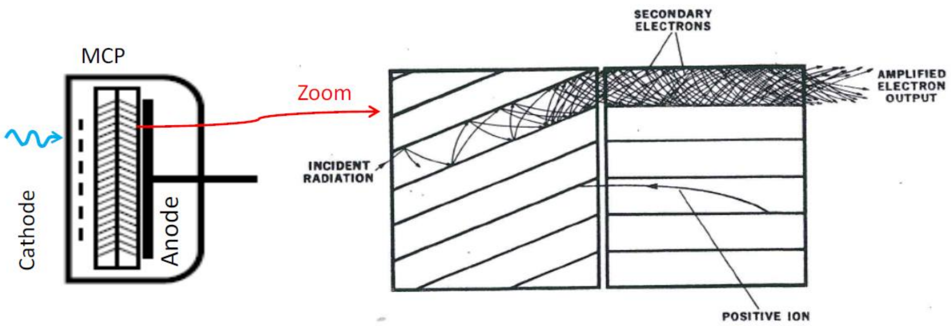
On the MCP, one electron can be emitted exactly horizontal to the tube and avoid any hit with the walls of the tube. This is not possible with the PMT. So if we change the number of impacts, it changes the gain. So for low noise applications the PMT is still better.

- Dark current is lower in CCM-PMTs than in dynode-PMTs, which collect also electrons from auxiliary input electrodes contaminated in the photocathode fabrication
- The **excess noise factor F>2 is significantly higher than dynode-PMTs**, because the statistical dispersion in the electron multiplication is clearly greater
- The inner layer resistance is in *GΩ* range, the current in this voltage divider is low *<1μA,* hence for avoiding nonlinearity the **mean output current must not exceed a few nA**. This sets a strict limit to the product of mean photon rate and PMT gain.
- The amplification of a pulse signal leaves a charge on the multiplier surface near to the output. A high charge modifies the electric field, impairing the amplification of the following pulses during a long recovery transient (discharge through the inner layer resistance, with time constant of milliseconds or more). **To avoid this, the product of multiplier gain and input pulse charge and/or repetition rate must be limited**
- Strong nonlinearities due to space charge may occur for high pulses and high gain

In conclusion, CCM-PMTs are

a) **well** suitable and provide very good performance for detecting **pulses with moderate repetition rate and small size** (down to single photons).
b) **NOT** well suitable for **many-photon-pulses** (e.g. for scintillation detectors of ionizing radiation) and for **stationary light intensity**.

- MCPs are implemented with small diameter D from 50μm to 5μm and the useful area (sum of the channel input sections) is ≈50 to 60% of the total plate area
- Each channel operates as an individual miniaturized CCM: the gain is optimized still with L/D ≈ 50
- To avoid ion feedback by bending the channel axis is not convenient for MCPs; the same principle is exploited by two MCPs with inclined channel axis, mounted in series with channel axis of the first and second MCP forming an angle



Most of the limitations that plague CCMs are relaxed for MCPs with illumination distributed on the cathode because:

a) Electrons emitted from the same position of the cathode do not enter all in the same microchannel; they are distributed over a group of facing channels in the MCP.

b) The perturbation of the voltage distribution in a channel affects the multiplication and collection of electrons just in that channel and closest neighbors, not farther.

It follows that:

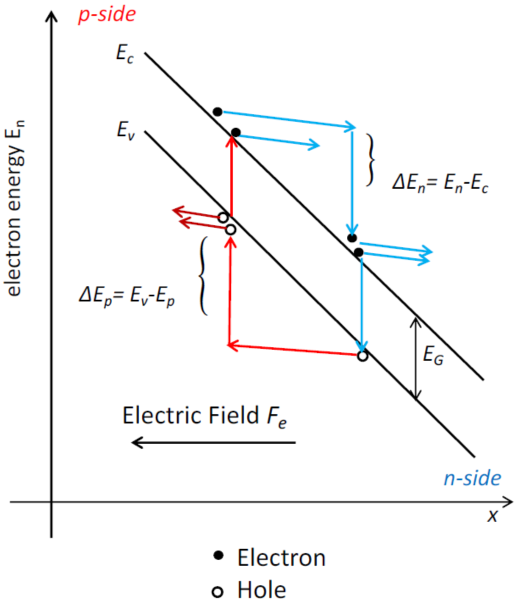
1) the limit to the output mean signal current is much higher; it is a small percentage of the **total** bias current of the MCP, not of a single microchannel

2) also many-photon optical pulses are correctly linearly processed, since the pulse photoelectrons are multiplied in parallel in different microchannels

- The statistical gain distribution of MCPs is similar to CCMs, significantly wider than for dynode-PMTs, with excess noise factor significantly higher F>2
- The dynamic response of MCPs is remarkably superior to that of dynode PMTs.
  The transit time T_b and its jitter T_j are remarkably shorter; in fast MCP types they are reduced down to $T_b \approx$1ns and $T_j \approx$ a few 10ps.
  Also the SER pulse-width T_w is shorter, down to $T_w \approx$ a few 100ps.

AVALANCHE PHOTODIODES

IMPACT IONIZATION IN SEMICONDUCTORS



- A free electron drifting in the field gains kinetic energy $\Delta E_n = En - Ec$
- Part of ΔE_n is transferred to lattice vibrations by scattering events
- Because of energy and momentum conservation, a ionizing collision can occur only when
  $$\Delta E_n > 1{,}5E_G$$
- Until reaching such ΔE_n the carrier travels without ionizing. The carrier multiplication thus has a dead-space; it is a **discontinuous statistical** process
- There is inherently a **positive feedback** loop in the process, because also holes can ionize by impact
- a cascade of ionizing collisions produces **avalanche multiplication** of carriers

In the image we have the CB and VB of a pn junction with an applied electric field. The idea is that if we increase a lot the electric field E, the electron in the CB can be accelerated by E ad if E is high, the energy the electron gains is so high that it can create another e/h pair due to impacts with the crystal. It is similar to the dynode in the PMT.
In order to create an e/h pair we need to gain an energy that is not the energy gap of silicon, but it is at least 1.5 the energy gap. This is because silicon is indirect gap.

If we gain an energy higher than this value, we can create a new e/h pair. So we are generating another e- that sees the same electric field, will be accelerated and eventually can create another carrier. But we are also generating an hole that goes in the opposite direction and can generate a pair. So it is a positive feedback mechanism. So it seems that from 1 or few electrons we can generate a lot of carriers, so a lot of current and a lot of signal.

However, we have a feedback, and we don't like this because we are adding noise (excessive noise factor). Moreover, theory of the APD is very complicated. Even if we have 1 electron that gains a lot of kinetic energy, and it reaches 1.4*Eg, nothing happens. So the process to generate a e/h process is not continuous, we need some time to gain energy to create new e/h pairs. So it is a discrete process. Having a discrete process makes it difficult to be analyzed and create a related model.

CONTINUOUS MODEL OF CARRIER MULTIPLICATION

We have to create a continuous process from a discrete one. If we are observing our phenomenon from a certain distance, with a time or space scale that is big enough to distinguish each ionization process, we can survive with an analog approach.
Since the ionizing process happens in the depleted region, we have to avoid thin depleted region. At this point the idea is to define an alpha, the ionizing coefficient for electrons, the probability density of ionization in the carrier path, and beta for the holes. **It is the probability for an e- to create another e/h pair after a dx space**. So we will have a probability lower than 1 that will increase with the energy acquired.

Starting from this simplification, we can define also the mean path between ionizing collisions.
The problem is that we have to add also **k**, which is the ratio between beta and alpha. It changes as a function of the material, it is almost 1 for indium-gallium-arsenide. Depending on k, the excessive noise factor completely changes.

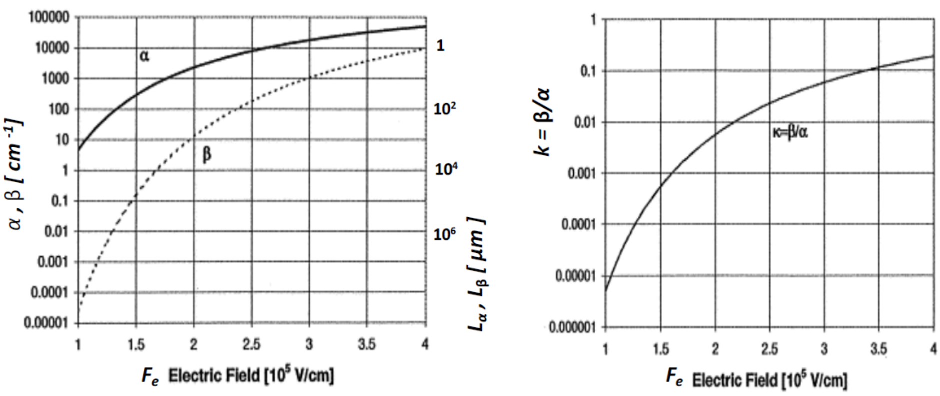
- The carrier multiplication can be analyzed with a **continuous statistical model**, based on the **average in space of the true discontinuous random process**.
- The continuous model provides a good approximation if the width of the multiplication region (high-field region) is definitely larger than the mean path between ionizing collisions. **The model is inadequate if the high-field region is very thin**, i.e. for width smaller than or comparable to the mean path between collisions.
- The model considers the probability of ionizing impact of a carrier as continuously distributed in space (i.e. it considers the average of many trials of carrier multiplication started by a primary charge).
- The **ionizing coefficients α for electrons and β for holes** are defined as the probability density of ionization in the carrier path; that is, for a carrier traveling over *dx* the probability of producing impact-ionization in *dx* is

  $\alpha\, dx$ for electrons and $\beta\, dx$ for holes
- The mean path between ionizing collisions thus is

  $L_\alpha = 1 / \alpha$ for electrons and $L_\beta = 1 / \beta$ for holes
- **The features of the multiplication process strongly depend on the relative intensity of the positive feedback, hence on the value of $k = \beta/\alpha$**, which is different in different materials: k<<1 in Silicon, k>1 in Ge and k≈1 in GaAs and other III-V

Ionization coefficients in silicon

Moreover, alpha and beta change as a function of the electric field. But in a pn junction the electric field is not constant in the depleted region, so alpha and beta change as a function of the position of the electron in the electric field.



- α and β **rapidly increase with the electric field F_e**. They can be described with good approximation by $\alpha = \alpha_o \exp\left(-\frac{F_{no}}{F_e}\right)$ and $\beta = \beta_o \exp\left(-\frac{F_{po}}{F_e}\right)$

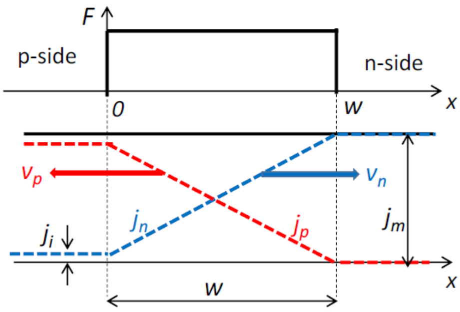
  In Silicon $\alpha_o = 3{,}8\ 10^6\ cm^{-1}$, $F_{no} = 1{,}75\ 10^6$ V/cm; $\beta_o = 2{,}25\ 10^7\ cm^{-1}$, $F_{po} = 3{,}26\ 10^6$ V/cm
- k is ≈ 0,1 at high electric field F_e and as F_e decreases k strongly decreases (because the dynamics of valence-band holes and conduction-band electrons are different)
- α and β **markedly decrease as temperature increases** (because stronger lattice vibrations drain more energy from carriers in the path between ionizing collisions)

Moreover, alpha, beta and k change also as a function of the temperature.

CARRIER MULTIPLICATION

We can use a PIN junction, so that the electric field is constant, so we can understand the current of e- and h in any point of the device.

- Even employing the continuous model, the complete mathematical analysis of the avalanche multiplication of carriers is quite complicated and will not be reported.
- However, the basic features of avalanche diodes can be clarified by analyizing a simple case. In a PIN junction with uniform and constant field higher than the impact ionization threshold, let us consider the stationary avalanche current due to the injection from the p-side of a small primary current of electrons j_i



Note that:

1. e-h pairs are generated, hence there are **both** electron and hole currents, even in case the ionization by holes be negligible (i.e. $\beta \approx 0$)
2. The total current is constant $j_m = j_n + j_p$
3. The p and n carriers of the avalanche form a **dipole-like mobile space-charge** (mostly p at p-side, mostly n at n-side) that adds a **field opposite to the junction field** (due to the fixed ion space charge)

We are interested in the total current in the device, not in the one of the single electron or hole.
We have to distinguish e- and h because as soon as the e- goes to the right, it's creating new pairs and so new electrons and holes, and the same for holes in the left. So to the right, the number of e- increases, while on the left the number of h is increasing. As soon as we have a lot of negative charge on one side and positive on the other, we are creating a dipole, and so an electric field that is in the opposite direction than the original electric field. Thanks to this charge distribution we will understand the final current in the device and also why the current is not diverging.

We are lucky since we have the equations. The total current in the device is the input current (original one) divided by 1-Ii, where Ii is the ionization integral. The formula holds if alpha = beta, so k = 1 (GaAs).

In the simplest case $\alpha = \beta$ (e.g. in GaAs) the equation is simply and we obtain:

$$\boxed{j_m = \frac{j_i}{1 - \int_0^w \alpha(x)dx} = \frac{j_i}{1 - I_i}}$$

$I_i = \int_0^w \alpha(x)dx$ is called **ionization integral** and has a clear physical meaning:

it is the probability for a carrier to have an ionizing collision in the path from x=0 to x=w

The current j_m is the primary current j_i amplified by the **multiplication factor M**

$$\boxed{M = \frac{j_m}{j_i} = \frac{1}{1 - I_i}}$$

This is the real formula, not a simplification. The ionization integral is the integral of the probability of the carrier to create ionizing impact in dx integrated from 0 to depleted region, so is the total probability of a carrier to generate one e/h pair in the whole depleted region.

Also in this case we can write the gain, called multiplication factor M. The problem is that M can diverge if Ii = 1. So if we generate at least one electron in the depleted region, the process is diverging. Infinite gain is not good because it is associated also to an infinite noise.
In order to have not infinite gain, the ionization integral Ii must be smaller than 1. Moreover, the formula is working for GaAs, but what about silicon, where alpha != beta, we have the following.

In cases with $\alpha \neq \beta$ the equation can still be integrated and the results can still be written in the form

$$j_m = \frac{j_i}{1 - I_i} \qquad \boxed{M = \frac{j_m}{j_i} = \frac{1}{1 - I_i}}$$

but the ionization integral I_i is now the integral of an effective ionization coefficient α_e

$$\alpha_e = \alpha \exp\left[-\int_0^w (\alpha - \beta) d\xi\right]$$

so that in this case

$$I_i = \int_0^w \alpha_e(x) dx = \int_0^w \alpha \exp\left[-\int_0^w (\alpha - \beta) d\xi\right] dx$$

Alpha is an equivalent alpha, and the ionization integral Ii can be computed as before, just alpha is changing. We will never compute alpha or Ii, we will stop on M.
As said, **M changes a lot as a function of the bias voltage**. In fact, if we change the bias voltage we change the electric field. Moreover, **M also depends on temperature**.

$$j_m = \frac{j_i}{1 - I_i} \qquad \boxed{M = \frac{j_m}{j_i} = \frac{1}{1 - I_i}}$$

- The ionization integral I_i in any case **strongly** depends on the **applied bias voltage V_a** and on the **temperature T**
- I_i is nil until the field F_e produced by V_a attains level sufficient for impact ionization
- Computations and experiments show that the **rise of M gets steeper as the high-field zone gets wider**. This is quite intuitive, since a wider zone corresponds to a higher number of collisions, which enhances the effect of the increased impact ionization probability due to an increase of the electric field

There is also another problem, since M gets steeper as the high-field zone gets wider. As soon as we increase the depleted region to collect more light, M gets steeper.

AVALANCHE BREAKDOWN

M can be divergent, and since alpha depends on the bias voltage, we can define the breakdown voltage, **which is the bias voltage at which the Ii = 1 and M diverges**.

- When the applied bias voltage V_a reaches a characteristic value V_B, the Ionization Integral $I_i \rightarrow 1$ and, according to the equation, M $\rightarrow \infty$ and $j_m \rightarrow \infty$

- ***V_B* is called Breakdown Voltage**; it is a characteristic feature of the diode, ruled by the distribution of the electric field F_e and by the dependance of α and β on the electric field F_e and on the temperature T

- V_B increases with the **temperature *T***. The increase is different in devices with different field profiles. It is anyway **strong**, some 0,1% per K degree.
  For Si it is about ≈*30 mV/K* in devices with V_B = *30 V* and ≈*900 mV/K* in devices with V_B = *300 V*.

Since the Vb has Ii = 1, and Ii depends on alpha, and alpha depends on temperature, also Vb depends on temperature.

In the real world it seems that we have either no current or infinite current. But it is strange to create something with infinite current, so we have to manage a formula that deals with infinite current and the real world where it doesn't exist. So we have to deal with the space charge effect, according to which we will reach a steady state value where the current is no more diverging.

- In reality, the **breakdown current is not divergent** and flows without requiring a primary injected current. In fact the current is self-sustaining, because of the positive feedback intrinsic in the avalanche ionization process.

- What keeps finite the avalanche current is the **feedback effect due to the mobile space charge**. The effect is negligible for $V_a < V_B$ (hence it is not taken into account in the former equations), but it is enhanced by the current rise at $V_a > V_B$ and reduces the electric field that acts on the carriers. The multiplication thus stabilizes itself at the self-sustaining level.

- For $V_a > V_B$ the avalanche current I_a increases linearly with V_a, so that an **avalanche resistance *R_a*** can be defined: $R_a = \Delta V_a / \Delta i_a$.

- In fact, ΔV_a produces a proportional increase of the electric field, which increases the impact ionization probability, hence the avalanche current. In turn, the current rise produces an increase of the space charge, which counteracts the effect of ΔV_a. The current thus rises until it brings back to self-sustaining condition the avalanche multiplication; that is, the current increase Δi_a is proportional to the voltage increase ΔV_a.

AVALANCHE PHOTODIODES

We want to be close to Vb but below it, so that Ii is not 1 and the current is not diverging.
The idea is to create a device that can work at a bias voltage below the breakdown in a stable way from the bias and temperature point of view. This is the difference in standard pn junctions and APD.

A photodiode biased at V_a **below the breakdown voltage V_B but close to it** provides **linear amplification** of the current by exploiting the avalanche carrier multiplication.

Such photodiodes with internal gain are called **Avalanche PhotoDiodes (APD);** they bear some similarity to PhotoMultiplier Tubes (PMT), but have remarkably different features

- The amplification gain is the multiplication factor M, which can be adjusted by adjusting the bias voltage V_a with respect to V_B
- Since V_B strongly depends on the diode temperature T, variations of T have effect equivalent to significant variations of the bias V_a. Therefore, for having a **stable gain M, the temperature of the APD must be stabilized**.
- The actual dependance of M on V_a can be fitted fairly well by an **empirical** equation

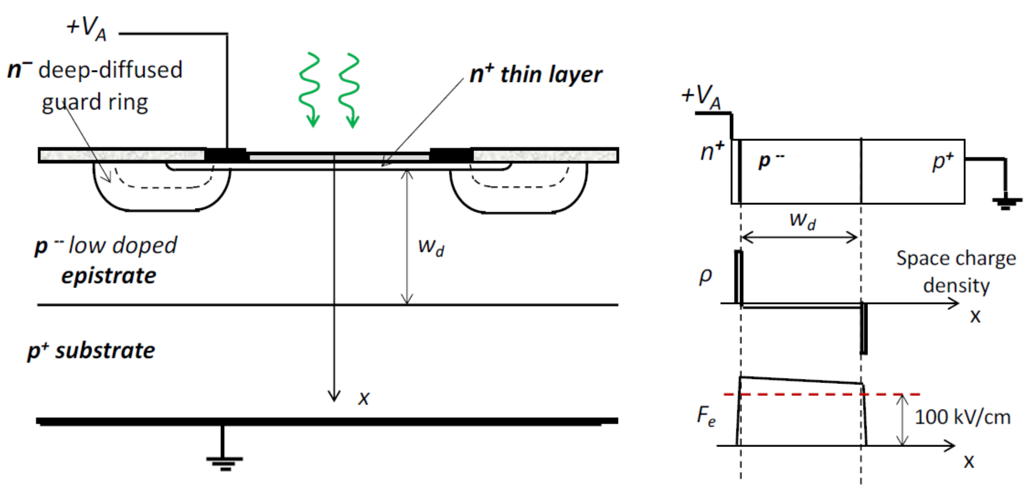
$$M = \frac{1}{1 - \left(\frac{V_a}{V_B}\right)^u}$$

- with exponent ***u*** that depends on the field profile (and on the type of semiconductor); it varies from 3 to 6, with higher values corresponding to wider high-field zone.

If we change the bias voltage we also change a lot the M according to the formula in the image. In fact, ripples on the bias voltage create a variation of M.

Evolution of the APD structure

Early attempts to develop APDs exploited **PIN structures modified** for operating at higher electric field (typically $F_e > 100$ kV/cm): more efficient guard-ring for avoiding edge breakdown; higher uniformity in material processing over the sensitive area; etc. The PIN structure, however, turned out to be unsuitable for APD devices.



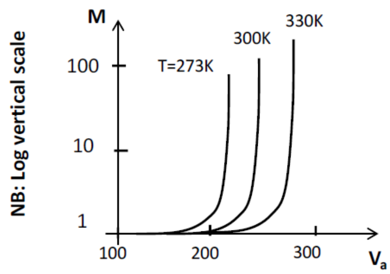
If we have a doping profile (to create e.g. the n region of the pn junction) and we change the radius of the doping profile we see the accumulation of electric field lines in the region where the doping region is not flat, so the electric field is high.

The problem is that we are trying to work with a bias voltage that is close to Vb but not above it. If in the flat region of the doping region we are just below Vb, at the corner where the electric field is higher we are above Vb.

So the idea is to create guard rings, diffusion region where the doping is much lower than in the flat part, so we compensate the fact that we have an accumulation of electric field with a lower electric field thanks to the presence of the guard ring. This is the difference between a standard pn junction, where we are very far from Vb because not of interest, and an APD.

PIN structure is unsuitable for APD devices

1. Even perfect p-i-n devices would have features not well suitable for operating as APD
2. Moreover, real p-i-n devices have unavoidable small local defects that rule out any prospect as APD.



- Even a perfect p-i-n diode would have multiplication factor M very steeply rising with the bias voltage V_a, because the depletion layer is wide (for obtaining high detection efficiency) and the high electric field zone covers it almost completely. **It would be extremely difficult to obtain a stable and accurately controlled gain M.**

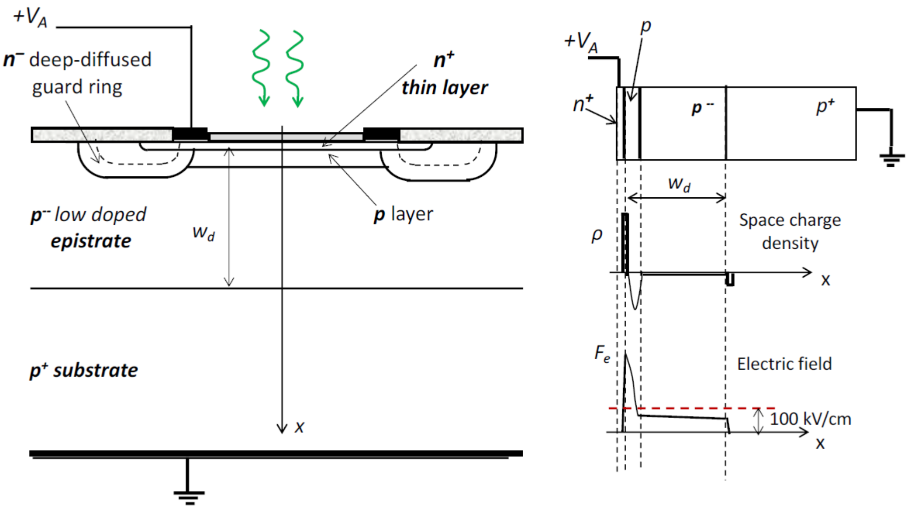
The evolution of the device design from PIN to Reach-Through APD structure was then driven by the insight gained in the PIN-APD failure.

In the graph we have the behaviour at different temperatures. We notice that M as reasonable values (100 in log scale), but it's very small compared to the PMT (millions). However, the real problem is the shape of the curves in the plot, because it's quite vertical, hence changing a little the bias voltage completely changes M.

The problem is that as soon as we increase the region where we have high electric field (depleted region), the behaviour of M becomes steeper and steeper. The problem is that we need a wide depleted region to collect light.
So the idea is to add a layer.

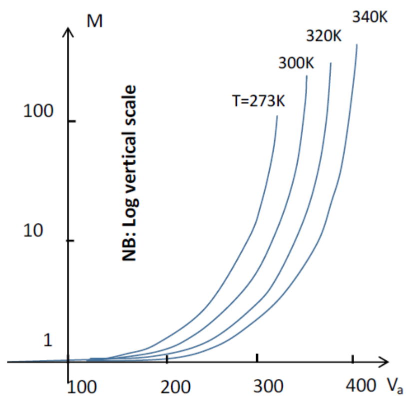
REACH-THROUGH Si-APD DEVICES (RAPD)
The region with very high electric field is limited to the p region, which is very small, but the depleted region is very large.
In the flat region (depleted region), the electric field can be reduced. I don't want high values there otherwise the behaviour of M is steeper and steeper, also because at a certain point we saturate the velocity and increasing more we just pay in terms of power consumption.
So in the depleted layer we have the saturated speed of the carriers.

Basic idea: to improve the structure by inserting a **thin layer with high electric field F_e** (where carriers undergo avalanche multiplication) beside a **wide depletion layer with moderate F_e** (where carriers just drift at saturated velocity)



Now we notice that the characteristic is not so steep, even if we still have a dependance on temperature. Moreover, the M value is not increasing, we are not changing its absolute value but its behaviour as a function of voltage. At the exam, taking M = 100 is good, because M = 1000 is impossible, and M = 10 is too low.

- The total depletion layer width of Si RAPDs in most cases is from 10 to 30µm, in order to obtain high detection efficiency up to 800-900nm wavelength (NIR edge)
- The width of the multiplication region (where F exceeds the ionization threshold) is much thinner, from 1 to a few µm
- Moderately steep rise of M with the bias voltage is obtained; the RAPD gain can thus be reliably controlled.
- The dependance of M on the device temperature is still remarkable and must be taken into account



The highest M obtained with Si-APDs is much lower than the gain level currently provided by PMTs. In the best cases M values up to about 500 are obtained; attaining M=1000 is out of the question

Avalanche statistics limits the APD gain

Avalanche multiplication is a statistical process → the APD gain has random fluctuations.

Let us denote by:

M the mean multiplication gain
σ_M^2 the gain variance and

In the multiplication, the fluctuations of the number of primary charges are not only amplified by M^2; they are **further enhanced by a factor *F>1*** called **Excess Noise Factor** (like for PMTs).

Input: primary carriers with
mean number N_p
variance $\sigma_p^2 = N_p$ (Poisson statistics)

Output: multiplied carriers with
mean number $N_u = M N_p$
variance $\sigma_u^2 = F M^2 \sigma_p^2 = F M^2 N_p$

We are neglecting what happens in the middle, we start with a signal and noise at the input and we look just at the output.

As for the PMT we can define an excessive noise factor.

The physical processes exploited for multiplying electrons in PMTs and in APDs are remarkably different and the detector gain has remarkably different features.

- In PMTs, the accelerated electron that hits a dynode is lost and the number of emitted secondary electrons fluctuates in a set of values that includes zero. The resulting mean number of carriers coming from the dynode is just the mean number of emitted secondary electrons and is definitely **higher than unity**.
- In APDs, the accelerated electron that undergoes a ionizing impact is not lost, it remains available for further impacts; the generation of a further electron (plus a hole) is statistical and the mean number of generated electrons is definitely **lower than unity**. The resulting mean number of electrons after the impact is **one plus the mean number of generated** electrons.
- In PMTs the gain is produced by an unidirectional sequence of events, the cascade of statistical multiplications at the various dynodes. Cascaded statistical processes can be well analyzed by known mathematical approaches (as the Laplace probability generating function)
- In APDs the **statistical process is much more complicated** than a simple cascade because of the intrinsic **positive feedback** in the impact-ionization. Rather than a cascade, it is a complex of interwoven feedback loops, each one originating from the other type of carrier (the hole in our case) generated in the impact.

In PMT the excessive noise factor F was lower than 2 and in some cases we could have considered it 1. This cannot be done with the APD because the physics of the device is completely different. In fact, the accelerated electron can generate at the best one pair, and not always. So the process is slower, which means higher noise.

Furthermore, in the APD we have also holes moving in the opposite direction, and so a positive feedback that creates problems in terms of noise.

For k << 1, that is the case of silicon, we can summarize the excessive noise factor F as below.

In Silicon with electric field intensity just above the ionization threshold, the situation is very favorable since the *F* degradation due to the positive feedback is negligible.

- The ratio of ionization coefficients is very small $k = \beta/\alpha < 0{,}01$
  → probability of impact ionization by holes much lower than that of electrons.
- the mean number μ of secondary electrons generated by the impact of an electron is **small** $\mu << 1$

The process can be analyzed as a cascade of electron impacts. By employing the Laplace probability generating function and numbering in sequence the impacts we get

$$\boxed{F = 1 + v_M^2 = 1 + \frac{1}{1+\mu} \approx 2}$$

***F=2* is the lowest possible *F* for Si-APDs and is achieved at low gain level. The conclusion is confirmed by experiments on carefully designed APD devices operating at *M<50*.**

For comparison, recall that ordinary PMTs routinely offer F < 2 at very high gain $M>10^5$.

In the best of the cases, F = 2, cannot be smaller. So for sure we cannot neglect F when considering an APD. Moreover, F = 2 can be obtained only with special devices and if M < 50. **As soon as we increase M, we also increase F**.
We want a gain so that the noise is negligible with respect to the noise of the next stage, that is the analog frontend (preamplifier).
The problem is that as soon as we increase the signal we are also increasing the noise, but not the noise itself, but the F number. At a certain point we will increase the noise more than how much we are increasing the signal, and the SNR drops down.

- Silicon with electric field just above the ionization threshold is a specially favorable case. In all other cases **the positive feedback in the avalanche process is remarkable**, it cannot be neglected and has detrimental effect on the variance of the APD gain.

- The fluctuation of the electrons generated in an impact is not only amplified by the further electron impacts in the subsequent multiplication path. The holes that are generated in the impact travel back and re-inject the fluctuation in a previous step of the multiplication path.

- This **back-injection of fluctuations enhances the excess noise factor *F***, with an efficiency that **increases with the *k* factor** (the relative ionization efficiency of holes versus electrons).

- **In Silicon the *k* factor markedly increases as the field is increased**. Therefore, *F* markedly increases as the bias voltage of the APD is raised for increasing the gain.

The positive feedback plays a key role. The situation is completely different if k = 1 or smaller. In silicon it is for instance almost 0.

A thorough mathematical treatment of the avalanche multiplication is quite complicated and beyond the scope of this course. We will just comment some results of treatments reported in the technical literature.

With some simplifying assumptions (uniform electric field; constant k value), it has been shown that the excess noise factor F **with primary current of electrons** is

$$F \approx M\left[1-(1-k)\left(1-\frac{1}{M}\right)^2\right]$$

- In cases with negligible positive feedback k=0, the equation confirms the result of the approximate analysis

$$F = 2-\frac{1}{M} \approx 2 \qquad \text{(since } M>>1\text{)}$$

- In cases with full positive feedback (i.e. equally efficient carriers, as in GaAs and other III-V semiconductors) it is k≈1 and F increases as M

$$F \approx M$$

- In cases with intermediate feedback level it is 0<k<1 and the equation specifies how F increases with M with rate of rise that increases with k. For instance:

  with k=0,01 at M=100 we get F≈3
  with k=0,1 at M=100 we get F≈12

When we want to compute the magnification to obtain, with a PMT we consider only the noise of the cathode and amplifier, while with the APD if we try to do this, we increase the noise of a factor M^2*F.

- The gain M of the APD is intended to **bring signal and noise of the detector to a level higher than the noise of the following circuits**, with the aim of attaining better sensitivity (smaller optical signal) than a PIN photodiode (limited by the circuit noise)
- However, when the voltage is raised for increasing M **also the variance of the gain fluctuations increases**. At some level M_{max} the effect of the gain fluctuations becomes greater than that of the circuit noise: increasing M beyond this level would be nonsense. This M_{max} limit depends on the actual case (actual APD and circuit).
- It is the **maximum factor F_{max} tolerable in the actual case that actually determines the M_{max} level**. In critical cases (typically InGaAs APDs, which have $F \approx M$) a fairly high value F_{max} turns out to be tolerable, even up to $F_{max} \approx 10$.

Thanks to the low k factor, Silicon devices have the lowest excess noise among APDs and achieve the highest gain levels.
Si-APD devices specially designed for low k have

F ≤ 2,5 up to M≈100
F ≤ 5 up to M≈500.

Ordinary Si-APD devices have fairly lower performance, i.e. typically

F ≤ 4 up to M≈100.

With Silicon, to get F < 2.5 we have to work with M = 100.

Conclusions

In III-V semiconductors (GaAs, InP, InAlAs, etc.) the ionization efficiencies of electrons and holes are equal ($k=1$) or at least comparable ($k\approx 1$). The positive feedback thus is very strong and F increases as M (see previous slides).

For InP-InGaAs and other III-V devices the useful gain range is fairly limited , typically:

$$F \lesssim 10 \text{ up to } M \approx 10$$

Nevertheless, InGaAs-APDs are in general preferred to Ge-APDs for detecting IR optical signals because they have lower dark-current (lower detector noise) and higher quantum detection efficiency, with cutoff to extended to longer wavelength (typically $\lambda \leq 1{,}7$ µm)

SINGLE-PHOTON AVALANCHE DIODES

SINGLE PHOTON COUNTING

With the APD we can detect a single photon, but the gain is very small compared to the PMT. Moreover, we have much stronger statistical fluctuations for the APD than the PMT, they can reach a value equal to the gain M.
So APD instead of PMT for single photon counting cannot be used. Or better, almost no for silicon APD and no for any other material.

APDs can detect smaller optical pulses than PIN diodes, thanks to the internal gain M.

However, the improvement of sensitivity is much lower than that brought by PMTs with respect to vacuum tube PDs. The reason is that in comparison to PMTs the APD gain M has

1. much lower mean value $\bar{M}$
2. much stronger statistical fluctuations, with relative variance that increases with $\bar{M}$

The **QUESTION** arises:

can we employ linear amplifying APDs instead of PMTs in single photon counting and timing techniques?

And the **ANSWER** is: **NO**!

More precisely, almost **NO** for silicon APDs and absolutely **NO** for APDs in other materials. In fact, we will now verify that only some special Si-APDs achieve single photon detection, although with marginal performance (detection efficiency lower than APD in analog detection; etc.), and other APD devices are out of the question.

APD FOR SPC?

- The APD output pulses due to a single primary carrier (single-photon pulses) are observed and processed accompanied by the noise of electronic circuitry, arising in the preamplifier and processed by the following circuits.
- A pulse comparator is employed to discriminate SP pulses from noise; pulses higher than the comparator threshold are accepted, lower pulses are discarded.
- The parameters of the set-up (rms noise; pulse amplitude; threshold level) should be ajusted to provide:
  1. Efficient **rejection of noise**, i.e. low probability of false detections due to the noise
  2. Efficient **detection of photon pulses**, i.e. high probability of detecting the SP pulses, which have variable amplitude with ample statistical fluctuations

We use a comparator to detect photons and not an amplifier because when we go to single photon counting we are in a totally digital approach, we don't have 1.5 photons. So we use a comparator to have digital pulses to collect.
Since it is digital, the output is digital, but the input is totally analog, since we have an amplification of the signal from the APD and then a comparator with a threshold. So the goal of the analog part that gives the digital output must efficiently reject the noise. So:

1. Efficient rejection of noise, so I want to set 1 in output only if I have a photon, not if I have noise (dark count). To solve the issue that the threshold is crossed without signal we can increase the threshold

2. Efficient detection of photon pulses.

Noise rejection in photon counting

- With noise amplitude having gaussian distribution (most frequent case) with variance σ_n (rms value), the **noise rejection threshold level must be at least** $\boldsymbol{N_{nr} \geq 2{,}5\, \sigma_n}$, in order to keep below <1% the probability of false detection
- We have seen that by employing an **optimum filter** for measuring the amplitude of detector pulses we get rms noise (in number of electrons)

$$\sigma_n = \frac{\sqrt{2C_L\sqrt{S_v}\sqrt{S_i}}}{e}$$

e =electron charge and typically: $C_L \approx$ 0,1 to 2pF load capacitance; $\sqrt{S_v} \approx$ 2 to 5nV Hz$^{-1/2}$ series noise; $\sqrt{S_i} \approx$0,01 to 0,1 pA Hz$^{-1/2}$ parallel noise

With high quality APD and preamp we get typically $\sigma_n \approx$ 40 to 120 electrons.
The noise rejection threshold required then is

$$N_{nr} \geq 2{,}5\ \sigma_n \approx 100 \text{ to } 300 \text{ electrons.}$$

Furthermore, M just higher than N_{nr} is not sufficient for having SP pulses higher than the threshold: we will see that **M much higher than N_{nr} is necessary**.

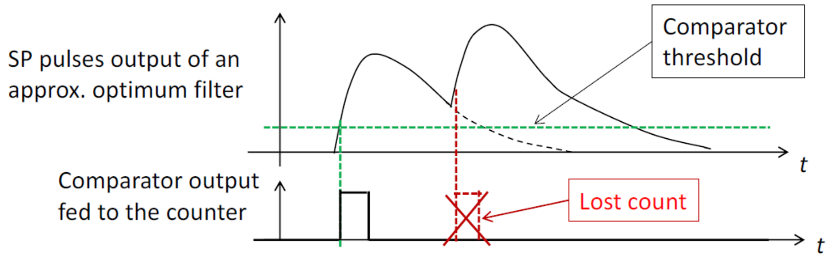
- We know that the optimum filter (and of course also an approximate optimum) is a low-pass filter and the output pulse has a width (i.e. a reciprocal-bandwidth) of some noise corner time constant T_{nc} . Since in our case T_{nc} ranges from 10ns to a few 100ns, the output pulses are fairly long and this brings drawbacks.

With a threshold larger than 2.5 or 3 sigma we are almost sure that we are not crossing the threshold with noise. For a SNR = 1 we need 40 to 120 electrons, and if we take 2.5 to maximize the noise we have from 100 to 300 electrons for SNR = 1. So it seems that we have to set a threshold of 100 e- to avoid the problem with noise. The APD can have a gain of 100, and also 300 is lower than 500 (maximum gain of the APD in special cases), and so the signal is greater than the threshold and noise is much smaller.

However, for some reasons, in the real world, **we need a M that is much higher than Nnr**.
The problem is that the formula in the image comes from the optimum filter theory, which is in the end a LPF, and in our case is between 10 and 100 ns. The point of the LPF is that when we want to detect single photons, we don't want to detect only one, but some of them, counting them. I have a threshold and to detect one photon we have to cross the threshold. To detect two photons I should cross the threshold two times, if the photons are not overlapped. For a 2 photon detection I have to cross the threshold 3 times at least (up, down, up). This is important because with a LPF we have a pulse of ligh that gives the response as in the image below, with a tail. If the second photon is close to the first one, there is a probability that the second signal is overlapped with the first one and the threshold is not crossed 3 times, it is like we have one single pulse and we loose one photon → count losses in photon counting.

The only way to solve the problem is to reduce the tail, making a faster system. But reducing the tail moves us to a sub-optimal filter, not the optimal one. Something that in the time domain is thinner has a bigger BW in frequency, and we are picking more noise, so the sigma has to increase, no more only 2.5.

- In photon counting **the finite width of the SP pulse causes count losses**. When the time interval between two photons is shorter than the output pulse width, pulse pile-up occurs (i.e. the two pulses overlap), the comparator is triggered only once and one count is recorded instead of two



- Photons occur randomly in time, hence the probability of pulse pile-up increases when the pulse width is increased.
- In conclusion, the percentage of lost counts increases as the pulse-width is increased. The width of the SP pulses should be minimized, in order to achieve efficient photon-counting with minimal percentage of lost counts.

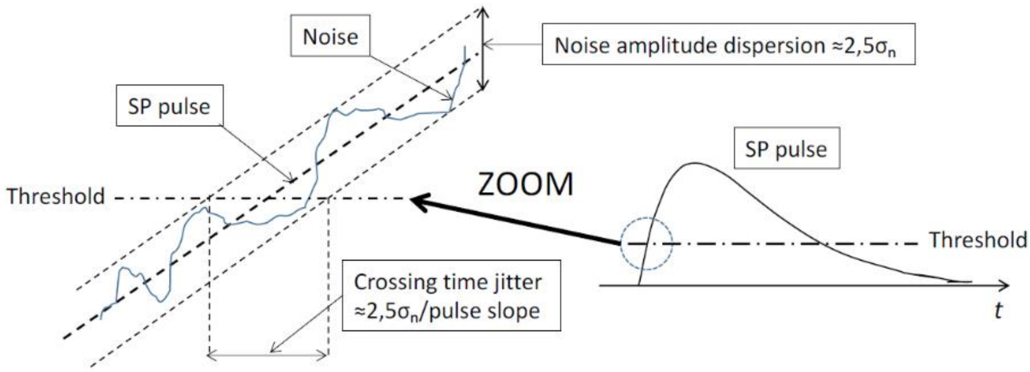
TIME JITTER IN PHOTON TIMING

We have the threshold and we need a clock, or a watch, that measures the time between the laser pulse and the crossing time of the threshold.
But are we sure that the crossing time is fixed? No, because superimposed to the signal we have noise and so we could have a jitter in the crossing time, so **the crossing time changes due to the noise**.

The jitter is in someway proportional to the variance of the noise. The problem is that we are proportional to the sigma of the noise, and this is the noise of the comparator for instance, not necessarily the one of the sensor.

To improve the situation without changing the comparator, we can change the derivative of the sensor, making a steeper response, so that the jitter influence is reduced. To increase the slope of the rising edge of the signal we have to increase the BW. Again, if we increase the BW we increase the noise.

- In photon timing, the arrival time of the pulse is marked by the crossing time of the threshold of a suitable circuit by the SP pulse.
- The noise causes **time jitter** (statistical dispersion) of the threshold crossing time
- A quantitative analysis is not reported here, but it is evident that the time jitter is proportional to the noise and **inversely proportional to the pulse rise slope**.
- A fairly long T_{nc} implies reduced pulse bandwidth and reduced slope of the pulse rise, hence **wide time jitter**.



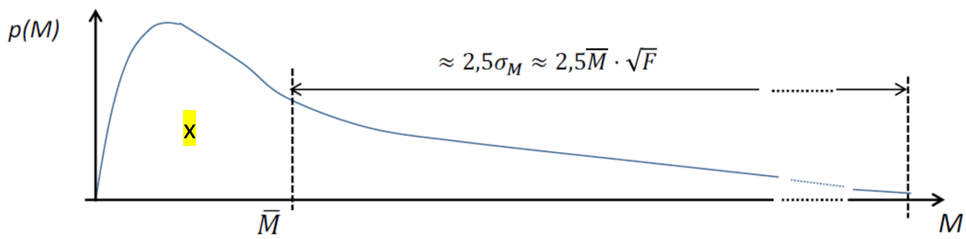
Photon counting and wide band electronics

For reducing count-losses and time jitter, we must process the APD pulses with filter bandwidth wider than the optimum filter. However, this implies higher noise, hence higher threshold level and higher gain required to the APD.

EFFICIENCY IN THE DETECTION OF SP PULSES

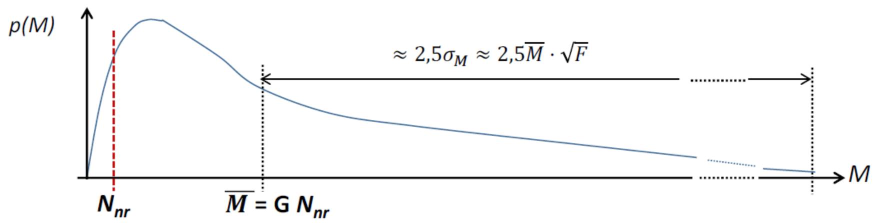
I expect that every photon is converted in 1 electron, and 1 electron in 150-300 electrons to cross the threshold. However, it is not true that the efficiency of the detector is 1 every time between photon and electron, so we don't have always the generation of an avalanche.
Moreover, we are not even sure that the electron that is amplified gives always 150 electrons in the avalanche, because we have the excessive noise factor F, we don't have always the same gain (the F formula in the image is not to be remembered).

- If the APD gain M were constant for all SP pulses, it would be sufficient to have M just higher than the noise rejection threshold level N_{nr} , but this is not the case.
- The gain M has strong statistical fluctuations, hence a high excess noise factor F>>1, which is directly related to the relative variance of M
$$F = 1 + v_M^2 = \frac{1 + \sigma_M^2}{(\overline{M})^2}$$
- The statistical M distribution thus has variance σ_M remarkably greater than the mean value $\overline{M}$
$$\sigma_M = \overline{M}\sqrt{F-1} \approx \overline{M} \cdot \sqrt{F}$$
- This implies that M has a strongly asymmetrical statistical distribution, with most of its area below the mean value $\overline{M}$ and a long "tail" above it



The point is that if in order to have a good SNR and neglect the noise we need 150 electrons starting from one electrons, 150 is not enough as a multiplication factor M, because the M has the behaviour as in the plot above. 150 is hence the average value.
So when the gain is lower than the average and lower than the threshold, we are actually not detecting the photon, and this happens in the x region. When the gain is lower than the threshold, the signal is lower than the threshold.

Sometimes, the amount of light that reaches the detector is really small and we have also to go fast. The solution is to choose a value of the gain that is much larger than the threshold. The point is that APD is not able to give too much gain.



- Therefore, with a mean gain $\overline{M}$ just above the noise rejection threshold a major percentage of the SP pulses is rejected. This downgrades the photon detection efficiency, i.e. the basic performance of the detector.
- In order to limit the reduction of detection efficiency due to the threshold, the mean gain $\overline{M}$ should be higher than the noise rejection threshold N_{nr} by a factor G>>1
- In the most favorable case (special Si-APD with optimum filtering), the value of $\overline{M}$ necessary for attaining the noise rejection threshold N_{nr} is near to the maximum available APD gain, but there is still some margin. In other cases (regular Si-APDs with wideband electronics) there is no margin at all.
- **CONCLUSION:** photon counting with linear amplifying APDs is possible only with special Si-APDs and with photon detection efficiency strongly reduced with respect to that obtained with the same APDs by measuring the analog current signal.

AVALANCHE DIODES ABOVE Vb

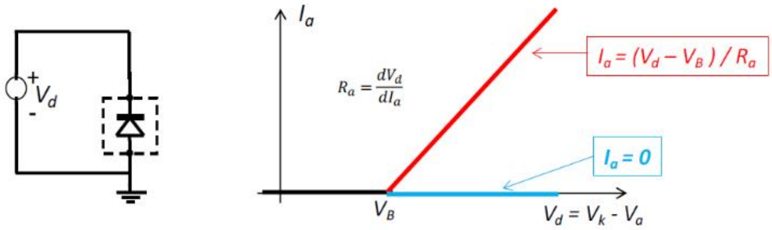
The APD is used above the breakdown voltage Vb. Since we know that positive feedback is a problem with an APD (when K = 1, F is much larger than when K = 0), we should avoid it. Instead, we can use the positive feedback to work above the breakdown voltage. At this point is suppose that every single photon that creates an electron in the depleted region will cause an avalanche, due to e/h pairs that generate other e/h pairs. So it seems an infinite charge with a single photon (even if it is limited due to space charge considerations).

- We have seen that the positive feedback inherent in the avalanche multiplication of carriers causes strong limitations to the internal gain of APDs in linear operation mode, thus ruling out the possibility of employing them instead of PMTs in single photon counting and timing.
- However, the positive feedback makes possible a radically different operation mode of some avalanche diodes, which working in this mode at voltage **above** the Breakdown Voltage V_B, turn out to be valid single-photon detectors.
- It is called **Geiger-mode** operation
  - Single photon switches on avalanche: macroscopic current flows
  - It's a triggered-mode avalanche: detector with "BISTABLE inside"
  - Avalanche is quenched by pulling down diode voltage $V_d \approx V_B$ (or below)
  - Diode voltage is then reset above the breakdown

- Such avalanche diodes, operating above the breakdown voltage in Geiger mode, generate macroscopic pulses of diode voltage and current in response to single photons. They are therefore called **Single-Photon Avalanche Diodes (SPADs)**.

The important thing is that the avalanche must be quenched. The name of the detector is SPAD.

SPAD I-V CHARACTERISTIC ABOVE Vb



The I-V characteristics shows a **bistable** behavior above breakdown $V_d > V_B$:

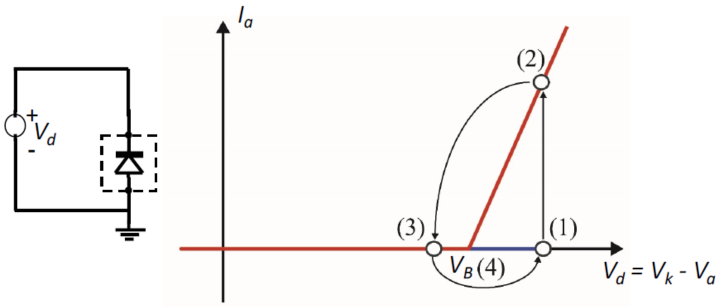
a) Without free carriers in the depletion region, I_A =0 above breakdown
b) at $V_d > V_B$ a self-sustaining avalanche can be **started even by a single free carrier** entering in the high field region at $V_d > V_B$. In this case I_A >0.

The higher the bias voltage above the breakdown, the higher the avalanche current. Therefore, the $\mathbf{\Delta V = V_d - V_B}$ is a key parameter: it is called excess bias or **overvoltage**.

To get the blue curve we should have no current, but it seems impossible because every electron in the depleted region will be accelerated and create another e/h pair. So the only way to have no current is to have no electrons, and this is the situation. If the depleted region is empty, nothing can be accelerated and there is no avalanche. Of course it is not a stable situation, since we might have thermally generated electrons. If we don't have them, the only way to generate an electron is to collect a photon.

Geiger mode operation

We are above Vb and two things can happen: either we have a thermally generated electron or the collection of a photon, and from point (1) we move to point (2). At this point, we have to bring down the voltage below the Vb to stop the avalanche to (3) and then bringing again the SPAD above Vb.



(1) Quiescent state: Bias voltage $\mathbf{V_d}$ above breakdown $\mathbf{V_B}$ (with excess bias $\mathbf{V_{exc}}$) is applied and no current flows

(2) Avalanche current flowing: it is triggered by a photon or noise

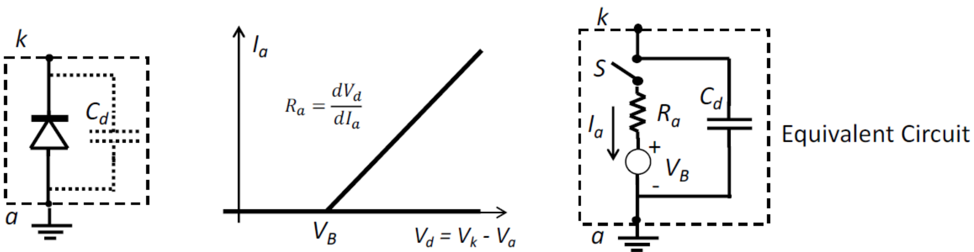
(3) Quenching: bias voltage $\mathbf{V_d}$ is lowered below the breakdown to stop the avalanche current flowing

(4) Reset: voltage across the junction is restored to the initial value

SPAD MAIN PROPERTIES

- In order to be able to operate in Geiger mode above the breakdown voltage, a diode should have uniform properties over the sensitive area: in particular, it must be free from defects causing local field concentration and lower breakdown voltage (the so-called microplasmas, due to metal precipitates, higher dopant concentration, etc.)
- Pulses are produced in SPADs also by the spontaneous thermal generation of single carriers in the diode junction and constitute a **dark count rate (DCR)** similar to that observed in PMTs. **Low DCR is a basic requirement** for an avalanche diode to be employed as SPAD.
- Various parameters characterizing the **detector performance strongly depend on the diode voltage**: probability of avalanche triggering, hence the photon detection efficiency; amplitude of the avalanche current pulse; dark count rate; delay and time-jitter of the electrical pulse with respect to the true arrival time of the photon; etc.
- **The breakdown voltage** depends on the structure of the device and on doping levels. V_B also strongly depends on junction **temperature**. At constant supply voltage V_d, the increase of V_B causes a decrease of excess bias voltage V_{ov}, impairing detector performance. Junction-temperature stability is very important.

EQUIVALENT CIRCUIT OF A DIODE ABOVE Vb



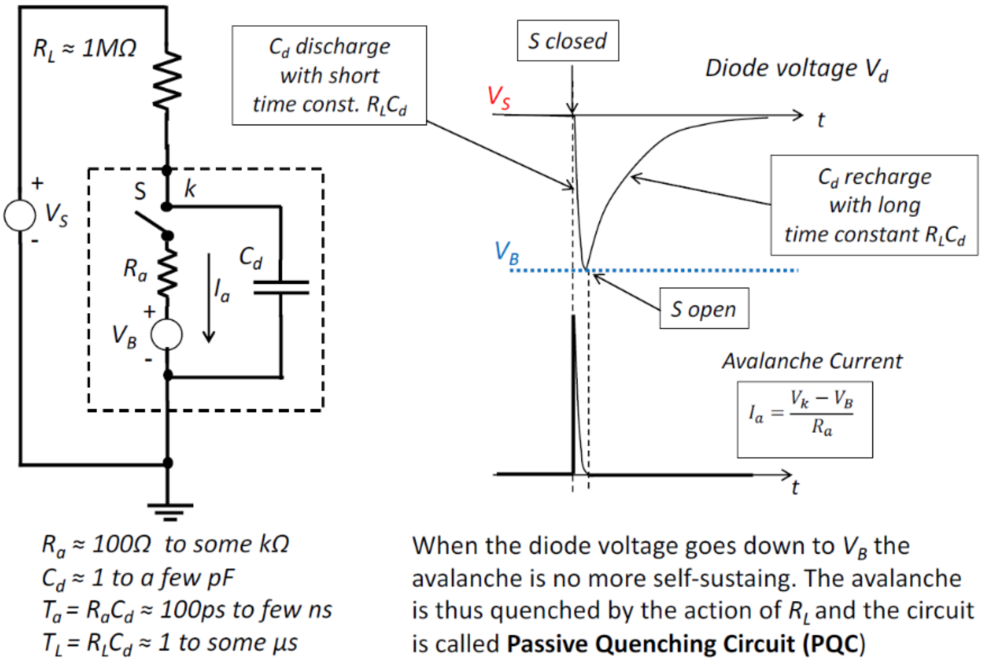
The equivalent circuit of the diode provides a quantitative understanding of the diode operation and confirms that the pulses observed correspond to single carriers generated in the device, spontaneously or by the absorption of single photons

- at $V_d > V_B$ the switch S can be closed or open; when it is closed, the avalanche current flows. At $V_d \leq V_B$ it is always open.
- **Closing the switch** is the equivalent of **triggering the avalanche** in the diode. Therefore, S is closed when a carrier injected or generated in the high field region succeeds in triggering the avalanche
- S then is open when the avalanche current is quenched (i.e. terminated) by the decrease of the diode voltage down to $\boldsymbol{V_d \approx V_B}$

We have a capacitance, a voltage generator that identifies the Vb, a series resistance, that is the space charge resistance, and a switch, because we have two conditions, ON or OFF. If there is current, the switch is closed, while if the voltage is below Vb the switch is open.

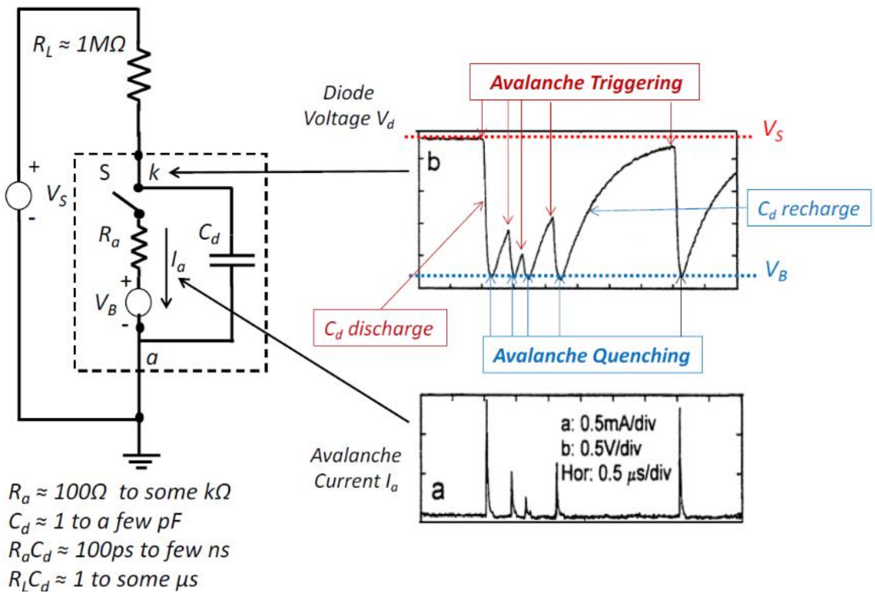
Passive quenching

We add a resistance on the top to quench the avalanche.



Passive quenching with repeated triggering

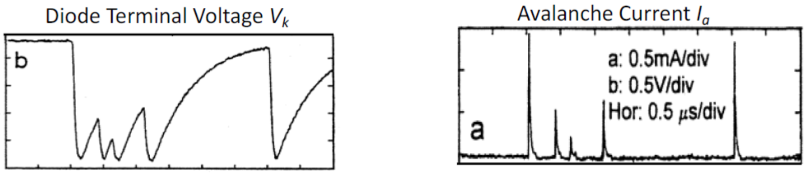
We notice that the shape of the pulse is not always the same. The rest is in fact slow since I have to charge the capacitance, and this is the problem of this application.



The point is that when we reach the blue Vb the avalanche stops and we open the switch because when the switch is closed, the bias on Ra and Vb cannot go below Vb because we have a Kirchhoff law. So how can we open the switch? The switch is open if we have no current or bias voltage below the Vb, so how is it possible that it works?

Trying to solve the KVL it is impossible that works, so the other possibility is that there is no current in the circuit.

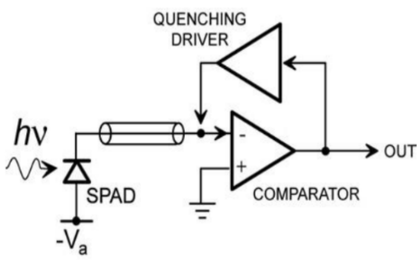
The current is composed by carriers travelling in a certain time, and R_L is a very big resistance, so the final current has to be in the order of 50-100 uA because if it is too low, the number of carriers is going down, and in the depleted region there are no more carriers, the avalanche is no more sustained. Until we have 1 carrier in the depleted region we still have an avalanche, but as soon as the current is so small that there is no more a carrier in the depleted region, the avalanche is no more sustained, and there is no current in the depleted region, not in the overall circuit.

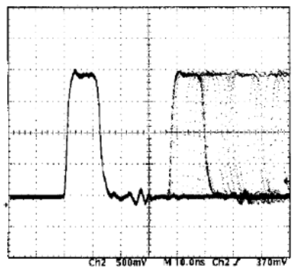


- In a passive-quenching circuit, after each quenching the diode voltage **slowly recovers from the breakdown voltage** V_B to the supply level V_S.
- In photon counting with a PQC, count losses are caused by the gradual recovery of the detection efficiency from nil to the correct level after each quenching.
- In photon timing with an avalanche diode in PQC, for photons arriving during a voltage recovery the arrival time measured on the electrical output pulse suffers **increased delay and time-jitter with respect to the operation** at the correct diode voltage. This effect progressively degrades the time resolution as the pulse counting rate is increased
- In conclusion, the application to photon counting and timing of avalanche diodes in Geiger mode with a PQC has very limited interest. It is restricted to favorable cases, that is **cases with low dark-count rate, low count-rate of background photons and low count-rate of the signal photons**

Active quenching

The problem with passive quenching is that if the rate of photons is high (kilo-count per second) we have problems in detecting the photons, we have different performances of the detector. So the solution is with active quenching circuits.
We use an electronics that senses the avalanche and has a quenching driver that stops the avalanche and resets the behaviour of the circuit. So the detector is in an off state for a fixed time, it is no more variable as in the passive quenching.





Output Pulses

by providing

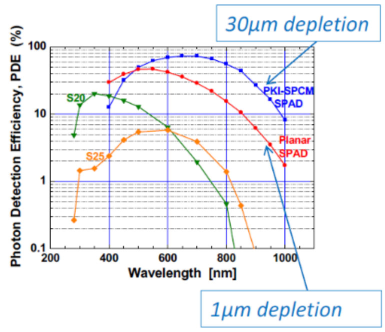
- short, well-defined deadtime
- high counting rate > 1 Mc/s
- good photon timing
- standard output

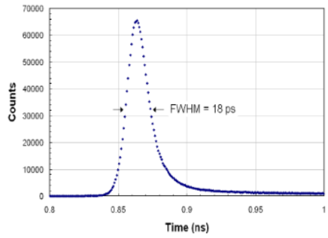
opened the way to SPAD applications

This solution gives us good timing and a standard output, not with pulses with amplitudes varying.

SEMICONDUCTOR SPADs VS PMTs

- **microelectronic advantages:**
  miniaturized, low voltage, etc.
- **improved performance:**
  higher Photon Detection Efficiency
  better photon timing
  comparable or lower noise





It is a very fast detector (ns) and with a good efficiency, better than the PMT. Moreover, they are used at relatively low voltages compared to the PMTs.

Challenges in SPAD development

Microelectronic Technology

- **Strict control** of transition metal contamination
  - ultra-clean fabrication process (defect concentration **< 10^9 cm^{-3}**)
  - suitable gettering processes **compatible** with device structure

Device design

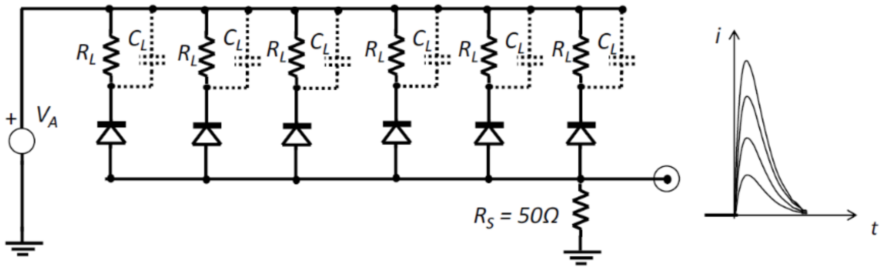
- **Electric field engineering**
  avoids BB tunneling and reduces field-enhanced generation, with impact on:
  → dark count rate
  → dark count decrease with temperature
  → photon detection efficiency
  → photon timing jitter

Front-end electronics

- **Low-level sensing of the avalanche current** → avoids or reduces trade-off between timing jitter and active area diameter
- **Application-specific** electronics

This device works in a bistable mode, photon or no photon, and so we don't have to have thermally generated electrons not to trigger an avalanche for many seconds to avoid unwanted avalanches.

SiPM – SILICON PHOTOMULTIPLIERS



This detector is a SPAD array where

- each pixel has an individual integrated quenching resistance $R_L \approx 100k\Omega$.
- each pixel has a very small individual load capacitance $C_L \approx 100\ fF$
- All pixels have a common ground terminal, connected to a low resistance external load, typically $R_S = 50\Omega$. The pixel currents all flow in this terminal, they are added

The detector pixels are thus

a) individually triggered by incident photons,

b) individually quenched by the discharge of the pixel capacitance

c) individually reset by the recharge of C_L with short time constant $R_L C_L \approx 10ns$

We still use PMTs because the PMT can give us something the SPAD cannot, that is the discrete increase in detected current, in the sense that counting the current we can detect how many photons are present simply checking the amplitude of the current. With the SPAD we cannot because we have an avalanche that is the same regardless of the number of photons impinging.

To solve this issue, we can use a lot of SPAD, so a SPAD array, and passive quenching. Passive quenching in theory has a long reset time, but here the capacitance is very small because everything is integrated, while the resistance remains almost the same. The problem of passive quenching is if we have two photons on the same detector during the reset time. If we have a lot of detector, the probability of having two photons on the same detector is very small. We can use millions of detectors in parallel in a 3mm x 3mm structure. The advantage is that if we have two photons on different sensors we have a current that increases in a discrete way with the number of detected photons as in the PMT.

The real problem is the noise. In fact, we have a lot of SPADs and the noise is increased a lot. So in terms of noise the PMT is still better for the same area occupation.

- The signal charge at the common output is proportional to the number of incident photons (at least as long as the light intensity on the detector is low enough to have negligible probability of more than one photon arriving on a pixel at the same time)
- Each pixel is a digital SPAD detector, but the pixel ensemble provides an analog information about the number of incident photons. The operation is indeed fairly similar to that of PMTs with microchannel plate multiplier. The detector was indeed conceived and is currently denoted as «Silicon PhotoMultiplier» SiPM.

With respect to PMTs, SiPMs offer various advantages

a) The typical properties of microelectronic devices (miniaturization; low voltage and low power; ruggedness; etc.)

b) remarkably higher detection efficiency, particularly in the red spectral range

c) operation insensitive to magnetic fields, which are detrimental for PMTs

However, SiPMs have also drawbacks with respect to PMTs

1. active area not as wide as PMTs
2. lower filling factor, with corresponding reduction of the photon detection efficiency
3. Fairly high dark current, that is, much higher dark current density over the active area

TEMPERATURE SENSORS

- Metallic RTDs: principle and fabrication
- RTD Electrical Signal
- Circuits for measurements
- Thermistors

METAL RTD PRINCIPLE

Principle:

- **Resistance** R_S of metal conductors **increases monotonically with temperature T**
- calibration of resistance versus temperature $R_S(T)$ is accurate and stable
- By measuring resistance variation ΔR_S we get the temperature variation ΔT

Linear behavior of $R_S(T)$ is a good approximation on wide T range for various metals

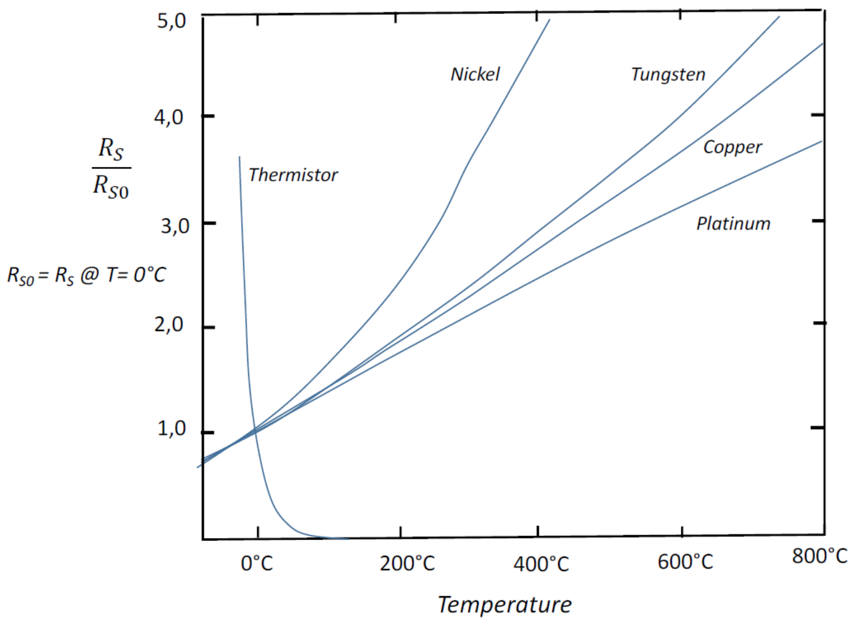
$R_S = R_0(1 + \alpha\Delta T)$ T_0 = reference temperature; $R_0 = R_S(T_0)$;

$\Delta R_S = \alpha\Delta T R_0$ $\Delta T = T - T_0$; $\Delta R_S = R_S - R_0$

α is called **temperature coefficient of resistance**.
α is around $\approx 4 \cdot 10^{-3}$ for metals currently employed in RTDs

Metal	α
Platinum Pt	$3{,}9 \cdot 10^{-3}$
Copper Cu	$4{,}3 \cdot 10^{-3}$
Tungsten W	$4{,}6 \cdot 10^{-3}$
Nickel Ni	$6{,}8 \cdot 10^{-3}$

It is a resistance that changes its value as a function of the temperature, and RTD has the advantage of being very linear. The value of alpha must be known for the exam, that is 4*10^3 (order of magnitude).



Metal RTD technology

Platinum has useful qualities:

- **Chemically inert and resistant to contamination**, hence stable properties
- R_S(T) **linear with very good approximation** from -200°C to about 500°C and with small deviation from linearity up to 800°C
- **small quantity of Pt necessary** in a RTD, cost is not high

Pt is the material of choice in many cases and is used in official metrology to define the International Practical Temperature Scale (from 13,81 K to 903,89 K).

Because of requirements for correct operation, the **RTD fabrication technology is not so simple** :

- The package must be compact and ensure good **thermal contact** of the resistor to the object measured and good **electrical isolation** from it
- Small size is required with R_0 **> some 10 Ω , typically** R_0 **=100 Ω**, in order to have to measure not very small ΔR_S. Thin wire wrapped in spiral on a support is used
- The mechanical structure must **avoid strain** of the metal wire due to thermal expansion or contraction: the **piezoresistive effect** would cause unwanted resistance variations and consequent errors in ΔT

From the device point of view, we have to make a really small sensor to measure the temperature on a small region, but at the same time the variation of the resistance has R0 as an initial value, so we want R0 big but in a small amount of space.

One of the problems is self-heating. To solve the problem we reduce the power dissipated on the sensor.

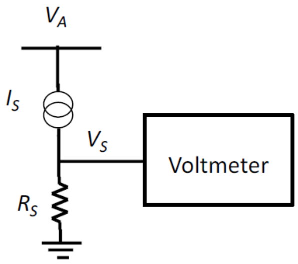
- RTD do not generate an electrical signal, a **power supply is necessary** to get current and voltage in the RTD
- Joule **self-heating** makes the RTD temperature T_S higher than the temperature T_a of the object measured; the difference $\Delta T_S = T_S - T_a$ increases with power dissipation P_S and sensor-to-object thermal resistance R_{th}.
- The maximum tolerable ΔT_S in a given RTD configuration sets a limit P_{Smax} to the power dissipated in the RTD, hence to the **maximum voltage** V_S on the RTD

$$P_S = \frac{V_S^2}{R_S} \qquad P_S \le P_{S,\max} \qquad \boxed{V_S \le \sqrt{R_S \cdot P_S}}$$

- The allowed voltage V_S on the RTD is fairly small: e.g. with $R_S \approx 100\ \Omega$ and limit P_{Smax} = 100µW, the voltage is limited to V_S <100mV.
- The **voltage variations** to be measured for small variations of temperature are a small fraction of V_S , i.e. they are **definitely small**.

RTD OPERATION AT A CONSTANT CURRENT

With a constant current, the variation of voltage is proportional to the variation of temperature. Normally, we are interested in very small variations of the temperature, and the sensitivity of the RTD in this configuration is not so high, so we can resort to a differential stage. We use a fixed resistance to remove the offset.



$$\Delta R_S = f(\Delta T) \approx \alpha R_0 \cdot \Delta T$$

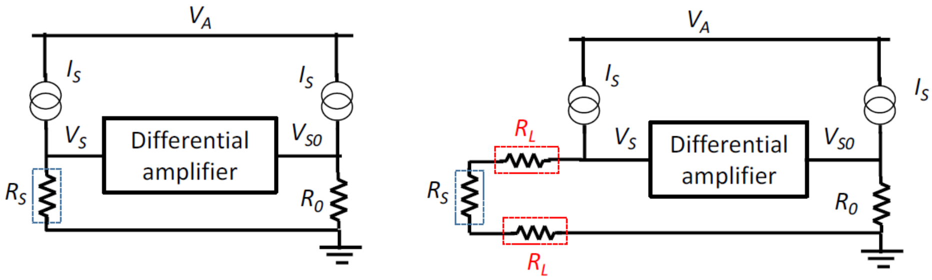
$$V_{S0} = I_S R_0$$

$$\Delta V_S = V_S - V_{S0} = I_S \cdot \Delta R_S = \\ = V_{S0} \frac{\Delta R_S}{R_0} \approx V_{S0} \alpha \Delta T$$

In modern electronics a simple approach is possible and practical thanks to the routine availability of current generators :

- R_S is biased with a **constant current** generator I_S ,
- voltage V_S on R_S is measured
- at any T, **V_S is exactly proportional to R_S** : the difference ΔV_S from measured V_S to reference voltage V_{S0} gives an accurate measure of ΔR_S
- ΔR_S is an accurately known function of $\Delta T = T - T_0$, in many cases approximately linear

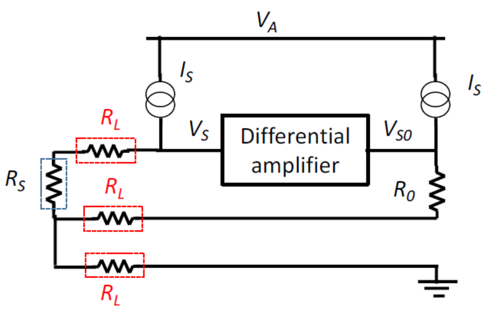
Differential signal at constant current



- Since ΔV_S is much smaller than V_S , it is advisable to include in the circuit a reference V_{S0} and take **directly differential measurements of ΔV_S** , instead of measuring V_S and then subtracting V_{S0}
- However, in various cases the RTD is placed on a measured object not near to the circuit, the **long connecting wires** have resistance R_L not negligible with respect to R_S and their **effect is significant** and must be taken into account
- In the simplest configuration, called «Two-wire-connection», the two wire resistances are in series with R_S and their voltage drop $2 I_S R_L$ is added to V_S , thus causing a significant error in the measured ΔV_S

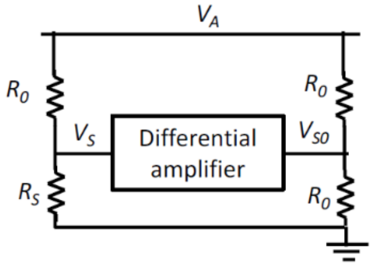
The problem is that normally the sensor is not near the electronics, so we might have a voltage drop over the cables. To solve this issue we add one more cable.

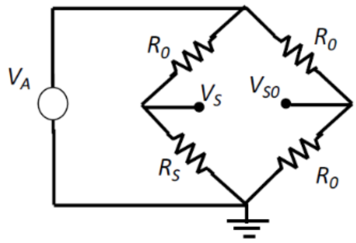
Remote RTD operation



- Errors in ΔV_S due to wire resistances R_L are avoided by a **«Three-wire-connection».** Both the reference arm and the RTD arm include in series a wire resistance R_L ; the third wire resistance R_L is inserted in the common return to the circuit ground

Wheatstone Bridge





- An alternative configuration, devised when current generators were not available, requires only resistors and due to its simplicity is still widely exploited
- A **voltage divider** is implemented by the $\boldsymbol{R_S}$ **of the RTD** in series with a **reference resistor** $\boldsymbol{R_0}$ and the variations of the divider output voltage corresponding to the variations of R_S are measured
- This is the principle of the **Wheatstone bridge**, invented in 1833 by Samuel Hunter Christie and popularized by Charles Wheatstone and usually drawn as sketched above at right

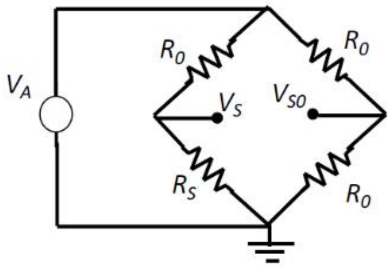
This configuration is easier and cheaper with respect to the one with two generators. Moreover, we have also a PS we can modulate with this configuration. To modulate the sensor, in fact, we have to modulate just one PS.

The signal we read is the differential one in between the two resistive partitions. We have to make the circuit linear.

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A \frac{R_0}{R_0 + R_0} = \frac{V_A}{2}$$

$$V_S = V_A \frac{R_S}{R_0 + R_S}$$



For **small resistance variation** $\Delta R_S < 0{,}05\ R_0$ the voltage variation ΔV_S is **approximately linear** with ΔR_S and can be computed by first-order development

$$\Delta V_S = \Delta R_S \left(\frac{dV_S}{dR_S}\right)_{R_S = R_0} = \frac{V_A}{4}\frac{\Delta R_S}{R_0} = \frac{V_A}{4}\alpha \Delta T$$

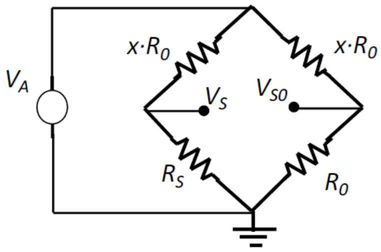
As soon as the left and right sides are the same, we can set R0. We use one single R0 because it's cheaper. On the top, instead of R0 we put x*R0 because we want to optimize the SNR and we don't know which is the resistive divider that optimizes it.

Then we take the derivative as a function of x and we get that is maximum for x = 0. For this value we maximize the sensitivity of our system.

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A \frac{R_0}{R_0 + xR_0} = \frac{V_A}{1+x}$$

$$V_S = V_A \frac{R_S}{xR_0 + R_S}$$



The Wheatstone bridge can be employed with **any ratio x** of the voltage divider, i.e. R_S can be in series with a resistor x·R_0 with any value of the factor x. However, it is intuitive and readily verified that **with x=1 the highest output ΔV_S** is obtained

$$\Delta V_S = \left(\frac{dV_S}{dR_S}\right)_{R_S=R_0} \Rightarrow \Delta R_S = V_A \frac{x}{(1+x)^2} \frac{\Delta R_S}{R_0}$$

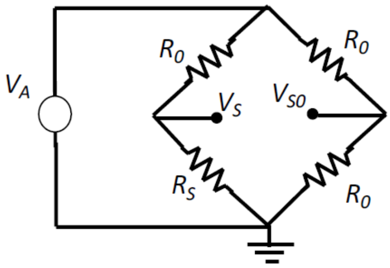
$$\max\left[\frac{x}{(1+x)^2}\right] = \frac{1}{4} \quad for \quad x = 1$$

Non linear operations

$$R_S = R_0 + \Delta R_S$$

$$V_{S0} = V_A \frac{R_0}{R_0 + R_0} = \frac{V_A}{2}$$

$$V_S = V_A \frac{R_S}{R_0 + R_S}$$



The cheap availability of integrated electronics for digital data processing and storage makes practical to extend the application of the Wheatstone bridge also to cases with **greater variations ΔR_S**, that have a **non-linear but known dependance** of ΔV_S on ΔR_S

$$\Delta V_S = V_S - V_{S0} = V_A \frac{R_0 + \Delta R_S}{2R_0 + \Delta R_S} - \frac{V_A}{2}$$

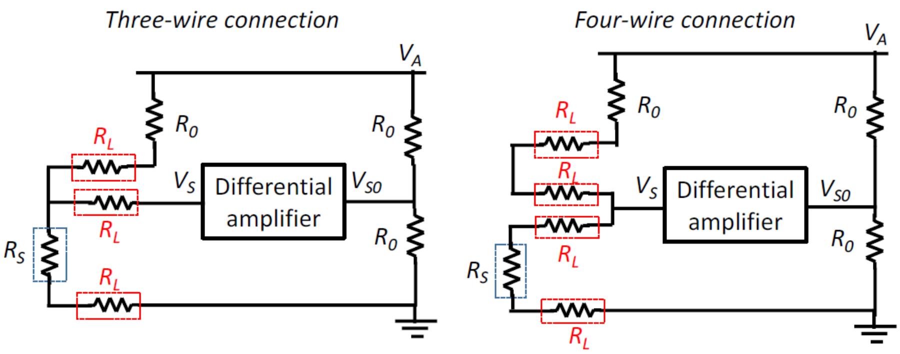
$$=$$

$$= \frac{V_A}{2} \cdot \frac{\frac{\Delta R_S}{2R_0}}{1 + \frac{\Delta R_S}{2R_0}}$$

From the theory point of view it’s easier to use the nonlinear operations.

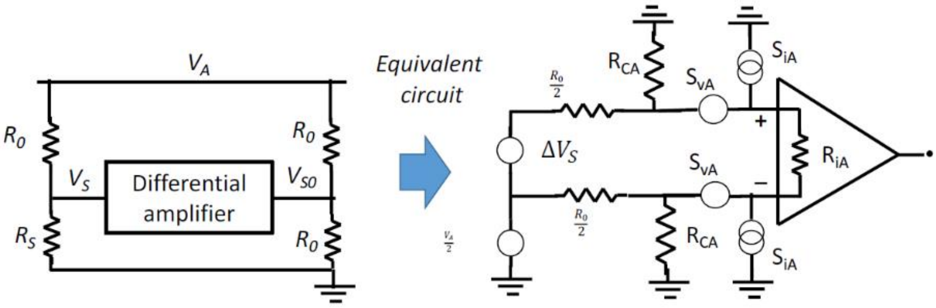
Remote RTD operation

If the sensor is far away from the frontend, and we are copying with small variations of resistance, the problem of the resistance of the wire is to be accounted for.
The variation of the temperature along the wire can create a variation of resistance which can be comparable with the signal we are reading. So we have to use a symmetric configuration to balance the situation.



- «**Two-wire connection**» causes error also in this case by adding $2R_L$ to R_S
- «**Three-wire-connection**» adds one R_L to the RTD and one to the balancing resistance R_0. The R_L of the connection to the differential amplifier is not compensated, but its effect is negligible because the current in it is negligible
- «**Four-wire-connection**» achieves complete simmetry between RTD arm and balancing arm, with complete cancellation of the errors due to wire resistances (and also cancellation of other minor thermoelectric effects caused by electrical current flowing in conductors with a temperature gradient)

RTD amplifiers



Since the **source resistance is low**, typically R_0=100 Ω:

- for the input differential resistance $\mathbf{R_{iA}}$ and the input-to-ground resistance $\mathbf{R_{CA}}$ **moderately high** values are sufficient
- the contribution of the input current noise generators is reduced, the input **voltage noise generators are dominant**

Since the differential signal ΔV_S is accompanied by a **high common mode signal $V_A/2$** :

- adequate **CMRR** is required **at the frequency of the supply V_A**, which can be selected at several kHz for reducing the 1/f noise contribution

R0 is typically small, so it is not difficult to have amplifier with high input resistance, it is not a hard constraint. The important thing is the CMRR. In fact we are interested in the differential input but both inputs are at half the dynamic range, so if we change the bias voltage we change the zero value and we must account for the common mode voltage variations.

THERMISTORS

- Commonly used temperature transducers called Thermistors are made of semiconductor ceramic materials, oxides of Cr, Mn, Fe, Co, Ni
- The dependence of thermistor resistance R on temperature is strikingly different from RTDs (see the plot in slide 29): strongly **nonlinear**, **decreases with increasing temperature** and the R values are **much larger** (some 100 kΩ at room temperature) and have much **greater relative variation**
- The resistance-temperature relationship can be described by the equation

$$R = \exp\left(\frac{B}{T}\right)$$

where **T is the absolute temperature** in Kelvin degrees, B is constant. B is called characteristic temperature of the termistor and usually ranges from 2000 K to 4000 K.
Making reference to the resistance value R_0 at a known reference temperature T_0 we get

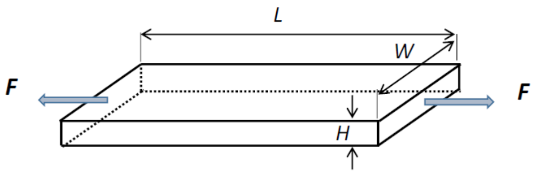
$$R = R_0 \exp\left[B\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]$$

It works as the RTD but the material is different and the behaviour is not linear, but an exponential decay. It is much smaller than the RTD, but the real important thing is that the behaviour is exponential, because also the small dynamic range is no more a problem.

- Thermistors can be made **much smaller** than RTDs.
- The smaller mass enables them to respond **more quickly** to temperature variations
- The **smaller size**, however, makes **less efficient the dispersion of the self-heating** power, which must be limited to low level
- The basic advantage of thermistors with respect to RTDs is **higher sensitivity**, i.e. larger relative variation ΔR/R for a given ΔT, which eases measurements of very small ΔT
- The main disadvantages are **lower accuracy and lower reproducibility** and strongly nonlinear characteristics, which limit the application of thermistors in automatic control systems

STRAIN GAUGES

STRESS AND STRAIN

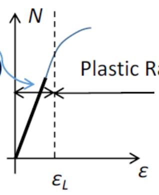


Metal bar with L = length; W = width; H = thickness; A= W·H cross section
F = pull force applied to the ends

- Stress **N = F/A force per unit area**
- ΔL = extension of L due to F
- **Strain $\varepsilon = \Delta L/L$** relative variation of L, measured in unit **$\Delta L/L = 10^{-6}$ = 1µstrain**

Up to the elastic limit ε_L (characteristic of material), strain ε is proportional to stress N.
For currently employed metals (steel, brass, etc.) the limit is $\varepsilon_L < 2\%$

Elastic Range
(reversible deformation)

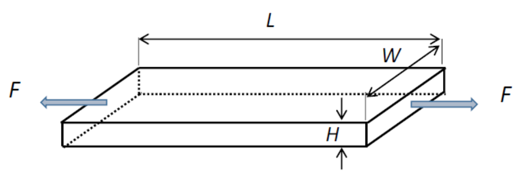


Plastic Range (permanent deformation)

$$\boxed{N = E\varepsilon}$$

***E* Young Modulus** (N/m²)

When we apply a force on a piece of metal, we are increasing or decreasing its length. The variation on length is proportional to the strength through the Young Modulus. At the same time, the orthogonal direction is decreasing according to the Poisson ratio.



In **elastic range,** a pull force F causes:

1) **Extension** of L proportional to stress: $\varepsilon = N/E$
   e.g. for steel $E \approx 200 \cdot 10^9$ N/m² = 200 GPa (1Pa = 1 Pascal=1N/m²)

2) **Contraction** of the section **dimensions** W and H proportional to the L extension ε

$$-\left(\frac{\Delta W}{W}\right) = -\left(\frac{\Delta H}{H}\right) = \nu \cdot \varepsilon$$

   ***ν* Poisson Ratio** (adimensional number)

   For most materials $\nu \approx$ from 0,25 to 0,4; for current metals $\nu \approx$ from 0,3 to 0,35

3) **Contraction** of the section **area** A= W·H (in absolute value)

$$\frac{\Delta A}{A} \approx \frac{\Delta W}{W} + \frac{\Delta H}{H} = 2\nu \cdot \varepsilon$$

PIEZORESISTIVE EFFECT

$$\boxed{R = \rho\frac{L}{A} = \frac{L}{\sigma A}}$$ R resistance; ρ resistivity; $\sigma = 1/\rho$ conductivity

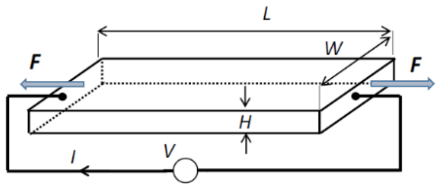
- Piezoelectric effect: in various materials a crystal lattice deformation changes the material resistivity, which contributes to the change of macroscopic resistance.
- **Strain changes the shape of the energy band curves** (energy vs momentum E-k), hence changes the electron effective mass m* and therefore the carrier mobility
- **Semiconductors have strong piezoresistive effect** and the dependence of conductivity on the strain is markedly nonlinear and strongly dependent on the semiconductor doping and on the temperature
- **Metals have small or moderate effect**, somewhat higher for Nickel and alloys than other metals. The dependence of conductivity on the strain N is fairly linear and a **piezoresistivity coefficient β** can be defined

$$\rho = \rho_o(1 + \beta N)$$

and the relative variation due to the piezoresistive effect can be described as

$$\frac{\Delta\rho}{\rho_o} = \beta N = \beta E \cdot \varepsilon$$

STRAIN GAUGE PRINCIPLE



$$R = \rho\frac{L}{A} = \frac{L}{\sigma A}$$ R resistance; ρ resistivity; $\sigma = 1/\rho$ conductivity

- In principle, a **Strain Gauge** (SG) is a long and thin metal slab (small cross section H << L and W << L) employed to measure the strain ε along its length L
- It is employed to measure strain in elastic range, without permanent deformation
- The relative variation of R is small (small elastic deformation and small or moderate piezoresistive effect) and can be **evaluated in first-order approximation***, i.e. denoting by subscript «o» the quiescent values without strain

$$\frac{\Delta R}{R_o} = \frac{\Delta L}{L_o} - \frac{\Delta A}{A_o} + \frac{\Delta\rho}{\rho_o} = \varepsilon + 2\nu\varepsilon + \beta E\varepsilon = \varepsilon(1 + 2\nu + \beta E)$$

* The finite small variation is computed as a differential

We want to see how the resistance changes as a function of stress. The relative variation of the resistance is the relative variation of the length, minus the relative variation of the area plus the relative variation of resisitivity. The relative variation of the length is the strain.

Gauge factor

$$\frac{\Delta R}{R_o} = \frac{\Delta L}{L_o} - \frac{\Delta A}{A_o} + \frac{\Delta\rho}{\rho_o} = \varepsilon + 2\nu\varepsilon + \beta E\varepsilon = \varepsilon(1 + 2\nu + \beta E)$$

- The conversion gain from strain ε to relative variation of the SG resistance R is called **Gauge Factor *G***

$$\boxed{G = \frac{\left(\frac{\Delta R}{R_o}\right)}{\varepsilon} = 1 + 2\nu + \beta E}$$

- Metal SG have small or moderate G:
  - G from 1,8 to 2,2 for most metals
  - G from 2 to 3,5 for Ni-Cu and Ni-Fe-Cr alloys
  - G≈12 for Nickel

  Since metals have about $\nu \approx 0{,}3$ a metal SG without piezoresistivity (i.e. with β=0) would have

  G≈ 1,6

  A comparison with the actual *G* values shows that the piezoresistivity contribution is significant, but it is not a big one

We have to remember at least the order of magnitude of G, which is from 1.8 to 2.2 for metals. There is a problem; we might have the same variation of resistance either due to the temperature or due to the strain.

To understand which is the variation of the sensor with respect to temperature we need to understand how the sensor is made.

DESIGN OF SG DEVICES

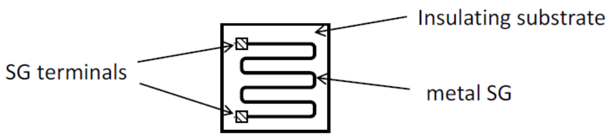
Conflicting requirements condition the design and fabrication of SG devices

a) **Requirement**: SG fastened to the sample under test for having the same strain
Solution: SG fastened onto a robust thin foil, which is then glued to the sample

b) **Requirement**: SG electrically isolated from the sample under test, for avoiding shunt effects due to conductive samples
Solution: SG supporting foil in insulating material

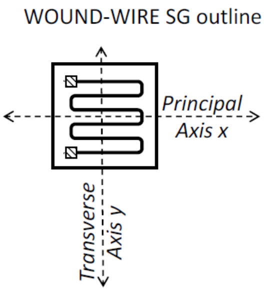
c) **Requirement**: small size of SG, for measuring the local strain and not strain averaged over a fairly wide area
Solution: limited size of the SG foil, as required by the case under test

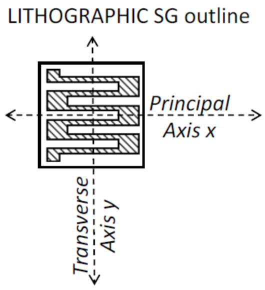
d) **Requirement**: not too small resistance of SG, for limiting measurement errors and uncontrolled parasitic effects (electrical contact resistance, etc.):
Solution: meander configuration of the resistor, in order to fit a long conductor length into the small area of the foil



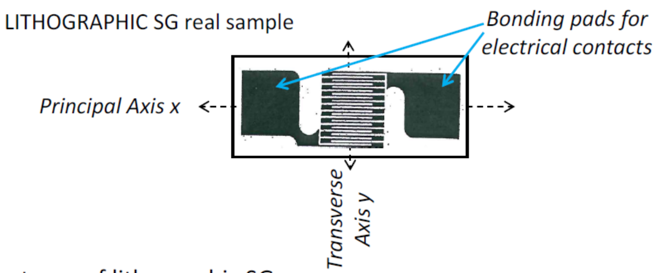
We want to create an electrical isolated potential, this is possible making it on an isolated foil that is then glued on the substrate to measure. Moreover, we want also small sizes, for measure local strains

- Old fashioned **wound-wire technology**: long thin **metal wire** wound in meander and fastened on insulating foil; strain measured on Principal Axis x (direction of meander long portions) with Principal Gauge Factor G_P
- Main drawbacks: a) sensitive also to strain along Transverse Axis y, though with a minor Transverse Gauge Factor $G_T \approx 0{,}05\ G_P$ b) moderate precision and reproducibility; well-matched SG samples are not available
- Modern **lithographic technology**: exploits lithographic technology (well developed in different scales for printed circuit boards and for integrated circuits) for finely designing SG of small size (1 cm and less) in a **very thin metal layer (from 2 to 10 μm) coated over an insulating foil**





with lithographic processes we can create precise shapes with small vertical transverse axis, so the device is less sensitive on that direction to the stress or strain. Moreover, the vertical part is also larger so that we reduce the resistance and so the variation of it.



Advantages of lithographic SG

- Wide transversal portions of the meander: their contribution to the SG resistance R is small so that the Transverse Gauge Factor is negligible $G_T < 0{,}001\ G_P$
- Small size < 1cm
- Metal conductor thickness in micron range gives high resistance per unit path; current R values are from 50Ω to 2kΩ, special SG are available with R>10kΩ
- Precisely defined device features, small tolerances in industrial production
- High reproduciblity: well matched SG devices are currently available
- The wide exposed surface area facilitates dispersion of heat generated in the resistor, thus reducing the SG self-heating

Electronic measurements with SG

As concerns the electronic measurement techniques, **Strain Gauges and Resistance Temperature Detectors (RTD) are essentially the same case**: small variations of a small resistance (typically a few hundred Ohms) must be measured with high precision.

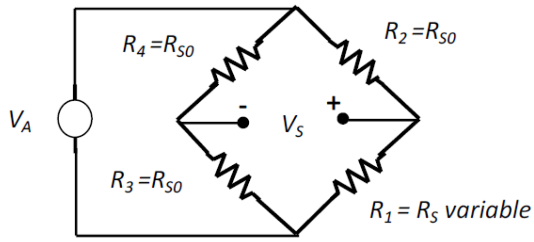
We will thus make explicit reference to the treatment of RTDs and add some notes about specific issues of SGs

- The SG resistance is $R_S = R_{S0} + \Delta R_S = R_{S0} + G\varepsilon R_{S0}$
- The Wheatstone Bridge with equal resistors (SG and other resistors with value R_{S0}) is a rational and widely employed solution. With small variations $\Delta R_S / R_{S0} << 1$ the signal V_S is proportional to the strain ε (as computed at 1st-order)

For a W-bridge with

- one SG of variable R_S
- three constant R_{S0}

$$\boxed{V_S = \frac{V_A}{4}\frac{\Delta R_S}{R_{S0}} = \frac{V_A}{4} G \cdot \varepsilon}$$



Let's suppose that for instance we want to measure 1 microstrain, and G = 2. Which is the maximum variation of temperature we can tolerate?

- The resistivity of metals increases with the temperature

  $\rho = \rho_0 + \Delta\rho = \rho_0 + \alpha\, \Delta T\, \rho_0$ (α temperature coefficient of the metal)

  for metals employed in SG it's around $\alpha \approx 4{\cdot}10^{-3}$ /K .
- Comparing R_S variations due to strain ε and to a temperature variation ΔT

  $$\left(\frac{\Delta R_S}{R_{S0}}\right)_N = G\varepsilon \qquad\qquad \left(\frac{\Delta R_S}{R_{S0}}\right)_T = \alpha\Delta T$$

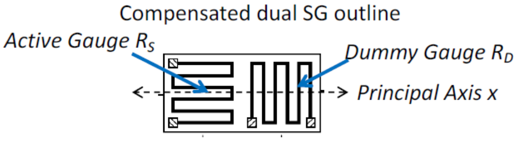
  we see that if the SG temperature T has an even small deviation $\Delta T = T - T_0$ from the reference temperature T_0 of the other resistors in the bridge, a remarkable error ε_T ensues. In fact, with $\alpha \approx 4{\cdot}10^{-3}$ /K and G≈2 the error is

  $$\varepsilon_T = \frac{\alpha\Delta T}{G} \approx 2 \cdot 10^{-3}\Delta T = 2000 \ \cdot \Delta T[in\ K] \ \ microstrain$$
- SG temperature deviations are often met in practice (e.g. SG working on motors or other structures with variable temperature) and produce unacceptable errors. Temperature effects in the SG cannot be avoided, but accurate **compensation** of their effect can be obtained by inserting in the Wheatstone bridge a properly devised **dummy gauge**

So it seems that we have to make a very precise measurement of the temperature to detect the strain, a precision of 1/1000 °C, but this is not feasible in reality.
With the Wheatstone bridge above, however, we measure both the temperature and the strain. So we need to resort to a different configuration.

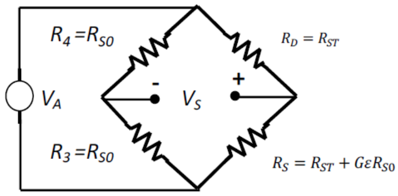
Compensation of temperature effects

Compensated dual SG outline



$R_S = R_{S0} + \alpha\Delta T R_{S0} + G\varepsilon R_{S0} = R_{ST} + G\varepsilon R_{S0}$

$R_D = R_{S0} + \alpha\Delta T R_{S0} = R_{ST}$

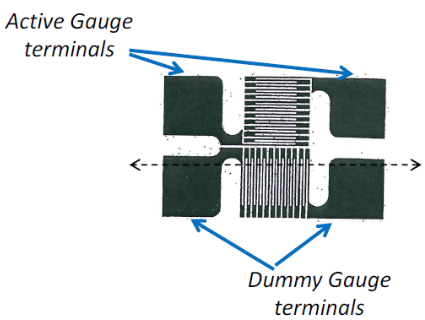


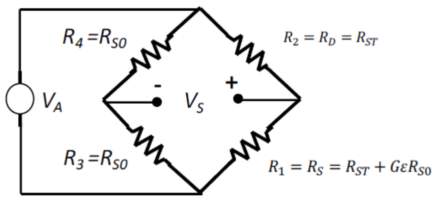
- Two identical gauges (Active Gauge and Dummy Gauge) are placed on the same foil with principal axes orthogonal
- The foil is glued to the structure under test, with principal axis of the active gauge in the direction of the strain to be measured
- The strain of the structure tested modifies the resistance R_S of the active gauge, but not the resistance R_D of the dummy gauge
- **Active and dummy gauge in close contact with the structure tested are kept at the same temperature** of the structure
- The power dissipation in the resistors must be limited by limiting the supply voltage V_A , in order to **limit the SG self-heating**

It is possible to have a sensor that is sensitive only to the temperature and not to the strain, but not viceversa. So what we do is to use the exact same sensor, placing it perpendicularly with respect to the sensing SG so that it compensate for temperature variations.
I don't have to know the temperature value because it is compensated using a dummy cell.

But where to place the dummy cell with respect to the sensor? R1 and R2 is the best since in this way the common mode voltage stays the same.

Compensated dual SG real sample



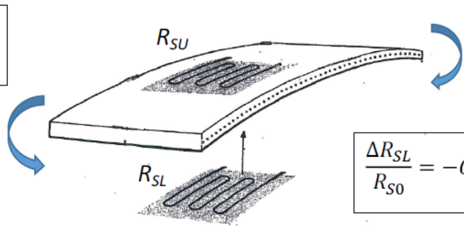


- In the bridge configuration shown (active gauge R_S inserted in R_1 position, dummy gauge R_D in R_2 position) the effects on the output voltage V_S of the temperature variation in R_S and R_D are compensated, hence V_S depends only on the strain ε
- Other alternative configurations of the bridge can be employed for compensation of the temperature effects; e.g. R_S inserted in R_1 position and R_D in R_3 position

MEASUREMENT OF BENDING

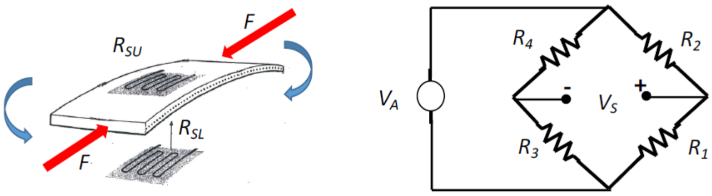
- With one Strain Gauge just a component of the strain is measured, the tensile or compressive strain in the direction of the SG principal axis.
- However, other strain components can be measured with more SGs rationally combined in the Wheatstone bridge
- Let's consider bending a long board with rectangular section (see the figure). The upper surface experiences a tensile strain $\boldsymbol{\varepsilon_B}$, the lower surface a symmetrical compressive strain **-** $\boldsymbol{\varepsilon_B}$. In fact, the strain linearly varies in the board section from $\boldsymbol{\varepsilon_B}$ to $-\boldsymbol{\varepsilon_B}$ and is zero in the mid, which is called «neutral plane»
- Let's consider to apply on the two surfaces of the board two matched SG (with equal resistance R_{S0} and Gauge factor G), denoted as R_{SU} on the upper surface and R_{SL} on the lower surface. Due to bending we get

$$\boxed{\frac{\Delta R_{SU}}{R_{S0}} = G\varepsilon_B}$$



$$\boxed{\frac{\Delta R_{SL}}{R_{S0}} = -G\varepsilon_B}$$

We place R1 and R2 in the same divider, and they are the sensor on the top and on the bottom of the bar, so one experiencing length increase, the other compression. If there is no bending, both sensors change of the same length. If we have bending, one is elongated and the other one is compressed.
But if the bending is something that I don't want to measure, what we have to do is to change configuration.



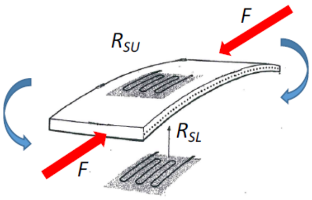
- With R_{SU} inserted in the bridge as R_1 and R_{SL} as R_3, we measure the bending strain $\boldsymbol{\varepsilon_B}$

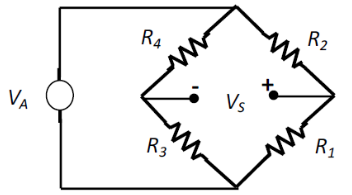
$$V_{SB} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} - \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = \frac{V_A}{2} G \cdot \varepsilon_B$$

- Let's consider now that a compressive force is added at the board ends: equal strain $\boldsymbol{\varepsilon_F}$ is added at the upper and lower surface, but the two SG have equal variation and the added contribution to the bridge output voltage is zero

$$\frac{\Delta R_{SU}}{R_{S0}} = \frac{\Delta R_{SL}}{R_{S0}} = G\varepsilon_F \Rightarrow V_{SF} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} - \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = 0 \Rightarrow \boxed{V_S = V_{SB} + V_{SF} = \frac{V_A}{2} G \cdot \varepsilon_B}$$

- In conclusion, by suitably employing two SG we can separately measure the net bending strain $\boldsymbol{\varepsilon_B}$ also in presence of an axial strain $\boldsymbol{\varepsilon_F}$





- On the other hand, with the same two SG we can also measure separately the net axial strain $\boldsymbol{\varepsilon_F}$ in presence of the bending strain $\boldsymbol{\varepsilon_B}$
- It is sufficient to change the configuration of the bridge. In fact, with R_{SU} inserted as R_1 and R_{SL} as R_4 we get

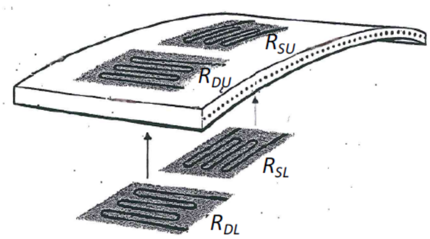
$$V_{SB} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} + \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = 0 \qquad\qquad V_{SF} = \frac{V_A}{4}\frac{\Delta R_{SU}}{R_{S0}} + \frac{V_A}{4}\frac{\Delta R_{SL}}{R_{S0}} = \frac{V_A}{2}G \cdot \varepsilon_F$$

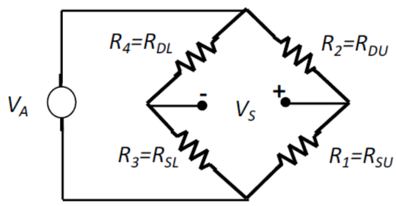
Therefore

$$\boxed{V_S = V_{SB} + V_{SF} = \frac{V_A}{2}G \cdot \varepsilon_F}$$

So the idea is to use other two dummy sensors to compensate for temperature.

- The measurements of $\boldsymbol{\varepsilon_B}$ and $\boldsymbol{\varepsilon_F}$ obtained with two matched SG as illustrated are correct only if the two SG are at the same temperature, but in many cases this is not achieved because the two SG are not in close proximity
- The drawback is avoided and the approach extended to all cases simply by
  a) employing dual compensated SGs instead of simple SGs and
  b) inserting in the bridge each dummy gauge in suitable position to compensate the associated active gauge





- Combinations of various SGs can be employed also for measurements in complex strain situations, i.e. with strain components in various directions, e.g. two-dimensional strain in aeronautical structures, such as aeroplane wings

SEMICONDUCTOR SG

- **Semiconductors such as Germanium and Silicon have very strong piezoresistive effect**. Strain Gauges in such materials thus provide large Gauge Factor G in the range from 100 to 300
- Magnitude and sign of the piezoresistive effect are governed by the type and level of doping. In p-type Silicon the effect is positive (tensile strain increases the resistivity) and in n-type silicon it is negative (tensile strain decreases the resistivity)
- **The effect is markedly dependent on the temperature**, with G decreasing significanly as the temperature is increased. A typical example is a reduction from G=120 at 10°C to G=105 at 65°C.
- **The Gauge Factor G is not constant as the strain is increased**, i.e. the gauge is not linear, with G decreasing significantly at moderately high strain. A typical example is a decrease from G=125 at 2000 microstrain down to G=100 at 4000 microstrain
- **The elastic range of these semiconductor materials is quite narrower** than that of metals, the elastic limit is typically at ≈4000 microstrain

In summary, semiconductor SGs suffer noteworthy limitations

- Response is **not linear**
- Response is **strongly dependent on the temperature**
- **Dynamic range is small**

but also offer remarkable features, such as

- **High Gauge Factor**, which provides high sensitivity: dynamic strains as small as 0,01microstrains can be measured
- **Small SG size <1mm**, which makes possible to measure highly localized strains, where a foil metal SG would be too large
- **Composite structures** including various resistors **can be fabricated** in a small region of the semiconductor crystal. The monolithic structure ensures equal temperature of the resistors and by selective doping it is possible to obtain different sign of piezoresistive effect in different resistors. Therefore, it is possible to devise SG configurations where the strain effects in different resistors inserted in a Wheatstone bridge collaborate to produce a voltage output, whereas the temperature effects are compensated

x x